Torsion of abelian varieties and Lubin-Tate extensions

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Abstract

We show that, for an abelian variety defined over a $p$-adic field $K$ which has potential good reduction, its torsion subgroup with values in the composite field of $K$ and a certain Lubin-Tate extension over a $p$-adic field is finite.

1 Introduction

Let $p$ be a prime number and $A$ an abelian variety over a $p$-adic field $K$ (here, a $p$-adic field is a finite extension of $\mathbb{Q}_p$). For an algebraic extension $L=K$, we denote by $A(L)$ the group of $L$-rational points of $A$ and also denote by $A(L)_{\text{tor}}$ its torsion subgroup. We are interested in determining whether $A(L)_{\text{tor}}$ is finite or not. The most basic result is given by Mattuck [Ma]; $A(L)_{\text{tor}}$ is finite if $L$ is a finite extension of $K$. Thus our main interest is the case where $L$ is an infinite algebraic extension of $K$. For this, Imai’s result [Im] is well known. He showed that $A(K(\mu_{p^\infty}))_{\text{tor}}$ is finite if $A$ has potential good reduction, where $\mu_{p^\infty}$ denotes the group of $p$-power roots of unity in a fixed separable closure $\overline{K}$ of $K$. Since the field $K(\mu_{p^\infty})$ is the composite field of $K$ and the Lubin-Tate extension over $\mathbb{Q}_p$ associated with a uniformizer $p$ of $\mathbb{Q}_p$, we naturally have the following question.

Question. Let $A$ be an abelian variety over a $p$-adic field $K$. Let $k_\pi$ be the Lubin-Tate extension associated with a uniformizer $\pi$ of a $p$-adic field $k$. Then, is $A(Kk_\pi)_{\text{tor}}$ finite?

In the case of Imai’s theorem ($k=\mathbb{Q}_p$ and $\pi=p$), the answer of the question is affirmative for potential good reduction cases, that is, the case where $A$ has potential good reduction. However, the question sometimes has a negative answer. For example, if $A$ is a Tate curve over $K$, $k=\mathbb{Q}_p$ and $\pi=p$, then $A(Kk_\pi)[p^\infty] = A(K(\mu_{p^\infty}))[p^\infty]$ is clearly infinite. We also have an example even for potential good reduction cases as given in Remark 2.10.

The aim of this paper is to give a sufficient condition on $k$ and $\pi$ so that the question has an affirmative answer for potential good reduction cases. Let $k, \pi$ and $k_\pi$ be as above. Let $q$ be the order of the residue field of $k$. We denote by $k_G$ the Galois closure of $k/\mathbb{Q}_p$. We put $d_G = [k_G : \mathbb{Q}_p]$ and denote by $e_G$ the ramification index of the extension $k_G/k$. We fix an embedding $\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}_p$.

Our main result is as follows (see Definitions 2.1 and 2.2 for some undefined notion).

Theorem 1.1. Let $A$ be an abelian variety over a $p$-adic field $K$ with potential good reduction. If $Nr_{k/\mathbb{Q}_p}(\pi)$ is not a $q$-Weil integer of weight $sd_G/t$ for any integers $1 \leq s \leq e_G$ and $1 \leq t \leq sd_G$, then $A(Kk_\pi)_{\text{tor}}$ is finite.

Applying Theorem 1.1 to the case where $k=\mathbb{Q}_p$ and $\pi=p$, we can recover Imai’s theorem. We should note that there is another generalization of Imai’s theorem which is given by Kubo and Taguchi [KT]. The main result of loc. cit. states that the torsion subgroup of $A(K(K^{1/p^\infty}))$ is
finite, where $A$ is an abelian variety over $K$ with potential good reduction and $K(K^{1/p^\infty})$ is the extension field of $K$ by adjoining all $p$-power roots of all elements of $K$.

For the proof of the above theorem, the essential difficulty appears in the finiteness of the $p$-power torsion part $A(Kk_\pi)[p^\infty]$ of $A(Kk_\pi)_{	ext{tors}}$. For this, we proceed our arguments in more general settings. We study not only abelian varieties but also (general) proper smooth varieties.

**Theorem 1.2.** Let $X$ be a proper smooth variety over a $p$-adic field $K$ with potential good reduction. Let $V$ be a $\text{Gal}(\overline{K}/K)$-stable subquotient of $H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p(r))$ with $i \neq 2r$. Assume that $V^{\text{Gal}(\overline{K}/L)} \neq 0$ for some finite extension $L/Kk_\pi$. Then $\text{Nr}_{k_\pi/\mathbb{Q}_p}(\pi)$ is a $q$-Weil number of weight $-(i - 2r)/h$ for some non-zero $h \in [-i + r, r] \cap \left( \bigcup_{s \in \mathbb{Z}, 1 \leq s \leq e_\sigma} (1/sd_\sigma)\mathbb{Z} \right)$. Moreover, $q^r \text{Nr}_{k_\pi/\mathbb{Q}_p}(\pi)^{-h}$ is an algebraic integer.

Applying Theorem 1.2 to the case where $k = \mathbb{Q}_p$, $\pi = p$ and $i$ is odd, we obtain [CSW, Corollary 1.6]. (Note that loc. cit. studies the vanishing of not only $H^0(\text{Gal}(\overline{K}/L), V)$ (as our result) but also $H^j(\text{Gal}(\overline{K}/L), V)$ for all $j$.) The assumption $i \neq 2r$ in Theorem 1.2 is essential as explained in the Introduction of [KT]. The key ingredients for our proof are the theory of locally algebraic representations (cf. [Se2]) and some “weight arguments” of eigenvalues of Frobenius on various objects. For weight arguments, we use $p$-adic Hodge theory related with Lubin-Tate characters and results on weights of a Frobenius operator on crystalline cohomologies (cf. [CLS], [KM], [Na]).

We hope our results can be useful for future studies in Iwasawa theory, for example, control theorems of Selmer groups for abelian varieties over certain $p$-adic extensions of number fields. In fact, arguments of [KT, Section 6] seem to be familiar with our results.

**Notation:** In this paper, we fix algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$, respectively, and we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. If $F$ is a $p$-adic field, we denote by $G_F$ and $U_F$ the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}_p/F)$ of $F$ and the unit group of the integer ring of $F$, respectively. We also denote by $F^{\text{ur}}$ and $I_F$ the maximal unramified extension of $F$ in $\overline{\mathbb{Q}}_p$, and the inertia subgroup $\text{Gal}(\overline{\mathbb{Q}}_p/F^{\text{ur}})$ of $G_F$, respectively. We set $\Gamma_F := \text{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)$. If $F'/F$ is a finite extension, we denote by $f_{F'/F}$ the residual extension degree of $F'/F$, that is, the extension degree of the residue fields corresponding to $F'/F$. We put $f_F = f_{F'/\mathbb{Q}_p}$. Finally, any $p$-adic representation of $G_F$ in this paper is of finite dimension.

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## 2 Proofs of main theorems

Our goal is to prove results in the Introduction. We often use $p$-adic Hodge theory. For the basic notion of this theory, it is helpful for the reader to refer [Fo1] and [Fo2]. In this paper, we normalize the Hodge-Tate weight so that the Hodge-Tate weight of $\mathbb{Q}_p(1)$ is one.

**Definition 2.1.** Let $q_0 > 1$ be an integer. A $q_0$-Weil number (resp. $q_0$-Weil integer) of weight $w$ is an algebraic number (resp. algebraic integer) $\alpha$ such that $|\iota(\alpha)| = q_0^w/2$ for all embeddings $\iota: \mathbb{Q}(\alpha) \rightarrow \mathbb{C}$.

**Definition 2.2.** Let $F$ be a $p$-adic field with residual extension degree $f = f_F$ and $F_0/\mathbb{Q}_p$ the maximal unramified subextension of $F/\mathbb{Q}_p$. We denote by $\varphi_{F_0}: F_0 \rightarrow F_0$ the arithmetic Frobenius of $F_0$, that is, the (unique) lift of $p$-th power map on the residue field of $F_0$. (1) Let $D$ be a $\varphi$-module over $F_0$, that is, a finite dimensional $F_0$-vector space with $\varphi_{F_0}$-semilinear map $\varphi: D \rightarrow D$. Then $\varphi^f: D \rightarrow D$ is a $F_0$-linear map. We call det$(T - \varphi^f | D)$ the characteristic polynomial of $D$. 

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(2) For a \( \mathbb{Q}_p \)-representation \( U \) of \( G_F \), we set \( D_E^{\text{cris}}(U) := (B_{\text{cris}} \otimes \mathbb{Q}_p U)^{G_F} \) and \( D_E^p(U) := (B_{\text{st}} \otimes \mathbb{Q}_p U)^{G_F} \), which are filtered \( \varphi \)-modules over \( F \). Here, \( B_{\text{cris}} \) and \( B_{\text{st}} \) are usual \( p \)-adic period rings. Note that we have \( D_E^{\text{cris}}(U) = D_E^p(U) \) if \( U \) is crystalline.

(3) Let \( S \) be a set of rational numbers. Let \( U \) be a potentially semi-stable \( \mathbb{Q}_p \)-representation of \( G_F \). Suppose that \( U|_{G_p} \) is semi-stable for a finite extension \( F' \) of \( F \) with residue field \( \mathbb{F}_p \). We say that \( U \) has \textit{Weil weights} in \( S \) if any root of the characteristic polynomial of \( D_E^{\text{cris}}(U) \) is a \( q' \)-Weil number of weight \( w \) for some \( w \in S \). (Note that this definition does not depend on the choice of \( F' \).

Let \( K \) and \( k \) be finite extensions of \( \mathbb{Q}_p \). Let \( q \) be the order of the residue field of \( k \), \( \pi \) a uniformizer of \( k \) and \( k_\pi \) the Lubin-Tate extension of \( k \) associated with \( \pi \). The following theorem is a key to the proof of our main results.

**Theorem 2.3.** Let \( S \) be a subset of \( \mathbb{Q} \setminus \{0\} \). Let \( V \) be a semi-stable \( \mathbb{Q}_p \)-representation of \( G_K \) with Hodge-Tate weights in \( [h_1, h_2] \). Assume that \( V \) has Weil weights in \( S \) and \( V^{G_K(\overline{\mathbb{F}}/L)} \neq 0 \) for some finite extension \( L/K_{\text{st}} \). Then

1. \( \text{Nr}_{k/\mathbb{Q}_p}(\pi) \) is a \( q \)-Weil number of weight \(-w/h\) for some \( w \in S \) and some non-zero \( h \in [h_1, h_2] \cap (\bigcup_{s \leq e_2} \{1/sd\_q\} \mathbb{Z}) \).
2. If the coefficients of the characteristic polynomial of \( D_{K_{\text{st}}}^1(V(\text{-}r)) \) are algebraic integers for some integer \( r \), then we can choose \( h \) in (1) so that \( q' \text{Nr}_{k/\mathbb{Q}_p}(\pi)^{-h} \) is an algebraic integer.

**2.1 Proof of Theorem 2.3**

In this section, we prove Theorem 2.3. We begin with some lemmas.

**Lemma 2.4.** Let \( (n_\sigma)_{\sigma \in \Gamma_K} \) be a family of integers. If there exists an open subgroup \( U \) of \( U_K \) with the property that \( \prod_{\sigma \in \Gamma_K} n_\sigma \sigma(x)^{n_\sigma} = 1 \) for any \( x \in U \), then we have \( n_\sigma = 0 \) for any \( \sigma \in \Gamma_K \).

**Proof.** Replacing \( U \) by a finite index subgroup, we may assume that the \( p \)-adic logarithm map is defined on \( U \). Then we have \( \sum_{\sigma \in \Gamma_K} n_\sigma \sigma(\log x) = 0 \) for any \( x \in U \) by assumption. Since \( \log U \) is an open ideal of the ring of integers of \( K \), we obtain \( \sum_{\sigma \in \Gamma_K} n_\sigma \sigma(y) = 0 \) for any \( y \in K \). Although the desired fact \( n_\sigma = 0 \) for any \( \sigma \in \Gamma_K \) follows from Dedekind's theorem [Bo, §6, no. 2, Corollaire 2] immediately, we also give a direct proof for this. Take any \( \alpha \in K \) such that \( K = \mathbb{Q}_p(\alpha) \) and let \( \Gamma_K = \{ \sigma = \text{id}, \sigma_2, \ldots, \sigma_c \} \) where \( c := [K : \mathbb{Q}_p] \). Then we have \( (n_{\sigma_1}, n_{\sigma_2}, \ldots, n_{\sigma_c}) X = 0 \) where \( X \) is the \( c \times c \) matrix with \((i,j)\)-th component \( \sigma_i(\alpha)^j \). Since \( \det X = \prod_{j>\sigma} (\sigma_j(\alpha) - \sigma_i(\alpha)) \neq 0 \), we obtain \( n_{\sigma_1} = n_{\sigma_2} = \cdots = n_{\sigma_c} = 0 \).

We denote by \( \chi_\pi : G_k \rightarrow k^\times \) the Lubin-Tate character associated with \( \pi \). If we regard \( \chi_\pi \) as a continuous character \( k^\times \rightarrow k^\times \) by the local Artin map with arithmetic normalization, then \( \chi_\pi \) is characterized by the property that \( \chi_\pi(\pi) = 1 \) and \( \chi_\pi(u) = u^{-1} \) for any \( u \in U_k \).

**Lemma 2.5.** Let \( E \) be a \( p \)-adic field and \( V \) an \( E \)-representation of \( G_K \). Assume that \( k/\mathbb{Q}_p \) is Galois, \( V \) is Hodge-Tate and the \( G_{K_{\text{st}}} \)-action on \( V \) factors through a finite quotient. Then, there exist finite extensions \( K'/K \) and \( E'/E \) with \( K', E' \supset k \) such that any Jordan-Hölder factor of \( (V \otimes E')|_{G_K} \) is of the form \( E'/(\prod_{\sigma \in \Gamma_K} \sigma^{-1} \circ \chi_\pi^r) \) for some \( r \in \mathbb{Z} \). Moreover, \( r_\pi \) is a Hodge-Tate weight of \( V \).

**Proof.** Replacing \( K \) by a finite extension, we may assume that \( G_{K_{\text{st}}} \) acts trivially on \( V \) and \( K \) is a finite Galois extension of \( k \). Since the \( G_K \)-action on \( V \) factors through the abelian group \( \text{Gal}(K_{\text{st}}/K) \), it follows from Schur's lemma that, for a finite extension \( E'/E \) of sufficiently large degree, any Jordan-Hölder factor \( W \) of \( V \otimes E' \) is of dimension 1. Our goal is to show that \( W \) is of the required form. We may assume \( E' = E \supset K \).

Let \( \rho : G_K \rightarrow GL_E(W) \cong E^\times \) be the continuous homomorphism given by the \( G_K \)-action on \( W \). Let \( \tilde{E} \) be the Galois closure of \( E/\mathbb{Q}_p \) and take any finite extension \( K'/K \) which contains \( \tilde{E} \). Since \( W \) is Hodge-Tate, it follows from [Se2, Chapter III, A.5, Theorem 2] that there exists
an open subgroup $I$ of $I_K''$ such that $\rho = \prod_{\sigma \in \Gamma_E} \sigma^{-1} \circ \chi_{E}^{n_{\sigma}}$ on $I$ for some integer $n_{\sigma}$. Here, $\chi_E: G_E \to U_E$ is the Lubin-Tate character associated with $\sigma E$ (it depends on the choice of a uniformizer of $\sigma E$, but its restriction to the inertia subgroup does not). Put $\tilde{\rho} = \prod_{\sigma \in \Gamma_E} \sigma^{-1} \circ \chi_{E}^{n_{\sigma}}$, considered as a character of $G_K''$. Replacing $K''$ by a finite extension, we may assume the following:

- $K''/\mathbb{Q}_p$ is Galois, $Gal(k_\pi/(k_\pi \cap K''))$ is torsion free and $\rho = \tilde{\rho}$ on $I_K''$.

Since $\rho|_{G_Kk_\pi}$ is trivial, we have that $\tilde{\rho}$ is trivial on $I_K'' \cap G_Kk_\pi = G((K'')_m)$. Hence, putting $N' = Gal((K'')_m/k_\pi)$, we may regard $\tilde{\rho}|_{I_K''}$ as a representation of $N'$. Put $N = Gal(k_\pi/k_m)$. Then $N'$ is canonically isomorphic to a torsion free finite index subgroup of $N \simeq U_k$, and thus we regard $N'$ as a subgroup of $N$.

Now we claim that $\tilde{\rho}|_{I_K''}$, regarded as a continuous character $N' \to \hat{E}^\times$, extends to a continuous character $\tilde{\rho}: N \to \mathbf{Q}_p^\times$. It follows from the theory of elementary divisors that we may regard $N = N_{tor} \oplus (\oplus_{i=1}^d \mathbb{Z}_p) \oplus \{0\} \oplus (\oplus_{j=1}^m \mathbb{Z}_p) = N'$ with some integer $m \geq 0$. Here, $N_{tor}$ is the torsion subgroup of $N$ and $d := [k : \mathbb{Q}_p]$. Hence it suffices to show that any continuous character $p^m \mathbb{Z}_p \rightarrow \mathbf{Q}_p^\times$ with $m > 0$ extends to $\mathbb{Z}_p \rightarrow \mathbf{Q}_p^\times$, but this is clear.

By local class field theory, we may regard $\tilde{\rho}|_{I_K''}$ and $\tilde{\rho}$ as characters of $U_{K''}$ and $U_k$, respectively. It follows from the construction of $\tilde{\rho}$ that we have $\tilde{\rho}(x) = \tilde{\rho}(N_{I_{K''}/k}(x))$ for $x \in U_{K''}$. In particular, we have

$$\tilde{\rho}(x) = \tilde{\rho}(\tau x)$$

for $x \in U_{K''}$ and $\tau \in \text{Gal}(K''/k)$. On the other hand, by definition of $\tilde{\rho}$ and the condition that $K''/\mathbb{Q}_p$ is Galois, we have

$$\tilde{\rho}(x) = \prod_{\sigma \in \Gamma_{K''}} \sigma^{-1}N_{K''/\sigma E}(x^{-1})^{n_{\sigma}} = \prod_{\sigma \in \Gamma_{K''}} \tilde{\sigma}^{-1}(x^{-1})^{n_{\sigma}}$$

(2.2)

for $x \in U_{K''}$ where $n_{\sigma} := n_{\sigma}$ if $\tilde{\sigma}|_E = \sigma$. We claim that $n_{\sigma} = n_{\sigma'}$ if $\tilde{\sigma}|_k = \tilde{\sigma}'|_k$. By (2.1) and (2.2), we have

$$\prod_{\sigma \in \Gamma_{K''}} \tilde{\sigma}^{-1}(x^{-1})^{n_{\sigma}} = \prod_{\tilde{\sigma} \in \Gamma_{K''}} \tilde{\sigma}^{-1}(x^{-1})^{n_{\tilde{\sigma}}}$$

(2.3)

for $x \in U_{K''}$ and $\tilde{\sigma} \in \text{Gal}(K''/k)$. Choosing a lift $\tilde{\sigma} \in \Gamma_{K''}$ for each element of $\text{Gal}(K'/k)$, we have a decomposition $\Gamma_{K''} = \bigcup_{\tilde{\sigma}} \tilde{\sigma}\text{Gal}(K''/k)$. Since $k_\pi/\mathbb{Q}_p$ is Galois, we see that $\text{Gal}(K''/k)$ acts on $\sigma \text{Gal}(K'/k)$ stably and this action is transitive. By Lemma 2.4, we know that the family $(n_{\sigma})_{\tilde{\sigma} \in \Gamma_{K''}}$ is determined uniquely by the restriction of $\prod_{\sigma \in \Gamma_{K''}} (\tilde{\sigma}^{-1})^{n_{\sigma}}$ to any open subgroup of $U_{K''}$. Hence the equation (2.3) gives $n_{\sigma} = n_{\sigma'}$ if $\tilde{\sigma}|_k = \tilde{\sigma}'|_k$ as desired.

For any $\sigma \in \Gamma_k$, we define $r_\sigma := n_{\sigma}$ for a lift $\tilde{\sigma} \in \Gamma_{K''}$ of $\sigma$, which is independent of the choice of $\tilde{\sigma}$ by the claim just above. Then we see $\tilde{\rho}(x) = \prod_{\tilde{\sigma} \in \Gamma_{K''}} \tilde{\sigma}^{-1}(x^{-1})^{n_{\sigma}} = \prod_{\sigma \in \Gamma_k} \sigma^{-1}N_{K''/k}(x^{-1})^{r_\sigma}$ for $x \in U_{K''}$. This implies

$$\tilde{\rho} = \prod_{\sigma \in \Gamma_k} \sigma^{-1} \circ \chi_\pi^{r_\sigma}$$

on $I_{K''}$. Now we define $\psi: G_K \rightarrow E^\times$ by $\psi := \rho \cdot (\prod_{\sigma \in \Gamma_k} \sigma^{-1} \circ \chi_\pi^{r_\sigma})^{-1}$. Then $\psi$ is trivial on $I_{K''}$ since $\rho = \tilde{\rho} = \prod_{\sigma \in \Gamma_k} \sigma^{-1} \circ \chi_\pi^{r_\sigma}$ on $I_{K''}$. Furthermore, $\psi$ is trivial on $G_{Kk_\pi}$ since $\chi_\pi$ and $\rho$ are trivial on $G_{Kk_\pi}$. Therefore, putting $K' = (K''_m \cap Kk_\pi)$, then $K'/K$ is a finite extension and $\psi$ is trivial on $G_{K'}$.

Finally, we note that $r_\sigma$ is a Hodge-Tate weight of $V$ by [Se2, Chapter III, A.5, Theorem 2]. This is the end of the proof.

**Lemma 2.6.** Let $E$ be a $p$-adic field and $V$ an $E$-representation of $G_K$. Assume that $k/\mathbb{Q}_p$ is Galois, $V$ is potentially semi-stable with Hodge-Tate weights in $[h_1, h_2]$ and the $G_{Kk_\pi}$-action on $V$ factors through a finite quotient. Then, there exists a finite extension $K'/Kk$ which satisfies the
following property: \( V|_{G_{K'}} \) is semi-stable and, for any root \( \alpha \) of the characteristic polynomial of \( D_{st}^{K'}(V) \), we have
\[
\alpha = a^{f_{K'}/k}, \quad a = \prod_{\tau \in \Gamma_k} \tau(\pi)^{-n_\tau}
\]
for some integers \((n_\tau)_{\tau \in \Gamma_k} \) such that \( dh_1 \leq \sum_{\tau \in \Gamma_k} n_\tau \leq dh_2 \). Here, \( d := [k : Q_p] \).

Proof. By Lemma 2.5, there exist finite extensions \( K'/K \) and \( E'/E \) with \( E', K' \supset k \) which satisfy the following:

\(- \quad V|_{G_{K'}} \) is semi-stable and any Jordan-Hölder factor \( W \) of \(( V \otimes E')|_{G_{K'}} \) is of the form \( E'(\prod_{\tau \in \Gamma_k} \sigma^{-1} \circ \chi_{K'\tau}) \) for some \( r_\tau \in [h_1, h_2] \). In particular, \( W \) is crystalline.

Replacing \( E \) by a finite extension, we may assume \( E' = E \). Now we take a root \( \alpha \) of the characteristic polynomial of \( D_{st}^{K'}(V) \), and choose \( W \) so that \( \alpha \) is a root of the characteristic polynomial of \( D_{cris}^{K'}(V) \).

To study \( \alpha \), we first consider the characteristic polynomial of \( D_{cris}^{K'}(E(\sigma^{-1} \circ \chi_{K'})) \) for \( \sigma \in \Gamma_k \).

We note that we have an isomorphism \( k(\sigma^{-1} \circ \chi_{K'})^{ss} \simeq k(\chi_{K'}^{ss}) \) of \( Q_p[G_K']-\text{modules} \) (here, “\( ss \)” stands for the semi-simplification of \( Q_p[G_K']-\text{modules} \)). In fact, for any \( g \in G_K' \), we have
\[
\text{Tr}_{Q_p}(g | k(\sigma^{-1} \circ \chi_{K'})) = \text{Tr}_{k/Q_p}(\text{Tr}_k(g | k(\sigma^{-1} \circ \chi_{K'}))) = \text{Tr}_{k/Q_p}(\sigma^{-1} \chi_{K'}(g)) = \text{Tr}_{k/Q_p}(\chi_{K'}(g)) = \text{Tr}_{Q_p}(g | k(\chi_{K'})).
\]

(Here, for a representation \( U \) of a group \( G \) over a field \( F \) and \( g \) in \( G \), we denote by \( \text{Tr}_F(g | U) \) the trace of the \( g \)-action on the \( F \)-vector space \( U \).) Therefore, we have
\[
\det(T - \varphi^{f_{K'}} | D_{cris}^{K'}(E(\sigma^{-1} \circ \chi_{K'}^{ss}))) = \det(T - \varphi^{f_{K'}} | D_{cris}^{K'}(k(\chi_{K'}^{ss}))[E:k]).
\]

To study the roots of \((2.4) \), we recall the explicit description of \( D_{cris}^{k}(k(\chi_{K'}^{ss})) \) (cf. [Con, Proposition B.4]. See also [Col, Proposition 9.10]). Let \( k_0 \) be the maximal unramified subextension of \( k/Q_p \). By definition, we have \( f_k = [k_0 : Q_p] \) and \( q = p^{f_k} \). Then \( D_{cris}^{k}(k(\chi_{K'}^{ss})) \) is a free \(( k_0 \otimes Q_p, k \))-module of rank one, and we can take a basis \( e \) of \( D_{cris}^{k}(k(\chi_{K'}^{ss})) \) such that \( \varphi^{f_k}(e) = (1 \otimes \pi)e \). We claim
\[
\det(T - \varphi^{f_k} | D_{cris}^{k}(k(\chi_{K'}^{ss}))) = \prod_{0 \leq i \leq f_k - 1} E^{e_i}(T)
\]
where \( E(T) = T^e + \sum_{j=0}^{e-1} a_j T^j \in k_0[T] \) is the minimal polynomial of \( \pi \) over \( k_0 \) and \( E^{e_i}(T) = T^{e_i} + \sum_{j=0}^{e_i-1} a_j T^j \). To show this, it suffices to show that the characteristic polynomial of the homomorphism \( 1 \otimes \pi: k_0 \otimes Q_p \to k_0 \otimes Q_p \) of \( k_0 \)-modules coincides with the right hand side of \((2.5) \). (Here, the \( k_0 \)-action on \( k_0 \otimes Q_p \) is given by \( a(x \otimes y) := ax \otimes y \) for \( a, x \in k_0 \) and \( y \in k \).)

We consider a natural isomorphism
\[
k_0 \otimes Q_p k_0 \cong \otimes_{j \in \mathbb{Z}/f_k \mathbb{Z}} k_{0,j}, \quad a \otimes b \mapsto (a \varphi^j(b))_j
\]
where \( k_{0,j} = k_0 \) for all \( j \). For \( 0 \leq s \leq f_k - 1 \), let \( e_s \in k_0 \otimes Q_p k_0 \) be the element which corresponds to \((\delta_{i,j})_{j \in \otimes_{j \in \mathbb{Z}/f_k \mathbb{Z}} k_{0,j}} \) where \( \delta_{i,j} \) is the Kronecker delta. Then \( (e_s(1 \otimes \pi^i) | 0 \leq j \leq f_k - 1, 0 \leq i \leq e - 1) \) is a \( k_0 \)-basis of \( k_0 \otimes Q_p k \). We see that the matrix of \( 1 \otimes \pi: k_0 \otimes Q_p k \to k_0 \otimes Q_p k \) associated with the ordered basis \( (e_0, e_{f_k-1}, e_0(1 \otimes \pi), \ldots, e_{f_k-1}(1 \otimes \pi)) \) is
\[
\begin{pmatrix}
O & O & \cdots & -A_0 \\
I_{f_k} & O & \cdots & -A_1 \\
\vdots & \ddots & \ddots & \ddots \\
O & \cdots & I_{f_k} & -A_{f_k-1}
\end{pmatrix}
\]
where \( I_{f_k} \) is the \( f_k \times f_k \) identity matrix and \( A_i \) is the \( f_k \times f_k \) diagonal matrix with diagonal entries \( a_1, \varphi(a_1), \ldots, \varphi^{f_k-1}(a_1) \). Now it is an easy exercise to check that the characteristic polynomial of this matrix is \( \prod_{0 \leq i \leq f_k - 1} E^{e_i}(T) \) as desired.
Now we note that roots of the characteristic polynomial of $D^{K'_{cr}}(k(\chi_{\pi}))$ are the $f_K'/k$-th power of those of $D^{K}_{cr}(k(\chi_{\pi}))$ since the latter describes the action of $\varphi^{f_K}$ but the former describes that of $\varphi^{f_K'/k}$. Furthermore, we also note that all the roots of the right hand side of (2.5) is a conjugate of $\pi$ over $\mathbb{Q}_p$. Hence, it follows from the claim (2.5) that any root of the characteristic polynomial of $D^{K'_{cr}}(k(\chi_{\pi}))$ is of the form $\tau(\pi)^{-f_K'/k}$ for some $\tau \in \Gamma_k$. On the other hand, for crystalline characters $\psi_1, \psi_2 : G_{\mathbb{Q}_p} \to k^\times$, we have a surjection $D^{K'_{cr}}(k(\psi_1)) \otimes_{K'_n} D^{K'_{cr}}(k(\psi_2)) \to D^{K'_{cr}}(k(\psi_1 \psi_2))$ induced from the natural map $k(\psi_1) \otimes_{\mathbb{Q}_p} k(\psi_2) \to k(\psi_1) \otimes_k k(\psi_2) = k(\psi_1 \psi_2)$. Here, $K'_n$ is the maximal unramified subextension of $K'/\mathbb{Q}_p$. In particular, roots of the characteristic polynomial of $D^{K'_{cr}}(k(\psi_1))$ is a product of those of $D^{K'_{cr}}(k(\psi_1))$ and $D^{K'_{cr}}(k(\psi_2))$. By this fact, we know that any root of the characteristic polynomial of $D^{K'_{cr}}(k(\chi_{\pi}))$ is of the form $\prod_{\tau \in \Gamma_k} \tau(\pi)^{-f_K'/k} \sigma^m$ with $\sum_{\tau \in \Gamma_k} m_{\tau} = r_\sigma$. By (2.4), the same holds for the roots of the characteristic polynomial of $D^{K'_{cr}}(W) = D^{K'_{cr}}(E(\sigma^{-1} \circ \chi_{\pi}))$. Therefore, since $\alpha$ is a root of the characteristic polynomial of $D^{K'_{cr}}(W) = D^{K'_{cr}}(E(\prod_{\sigma \in \Gamma_k} \sigma^{-1} \circ \chi_{\pi}))$, we have

$$\alpha = \prod_{\tau \in \Gamma_k} \tau(\pi)^{-f_K'/k} \sigma^m$$

with $\sum_{\tau \in \Gamma_k} m_{\tau} = \sum_{\sigma \in \Gamma_k} r_\sigma =: R$. (Here, $n_{\tau} = \sum_{\sigma \in \Gamma_k} n_{\sigma} \sigma^m$.) We note that $R$ is an integer such that $dh_1 \leq R \leq dh_2$ since we have $r_\sigma \in [h_1, h_2]$. This completes the proof. \hfill $\square$

We need the following two standard lemmas which describe inclusion properties of two Lubin-Tate extensions.

**Lemma 2.7.** Let $k_2/k_1$ be a finite extension of $p$-adic fields with residual extension degree $f$. For $i = 1, 2$, let $\pi_i$ be a uniformizer of $k_i$ and $k_1, k_2$ be the Lubin-Tate extension associated with $\pi_i$. Then we have

1. We have $N_{k_2/k_1}(\pi_2) = \pi_1^{f_k}$ if and only if $k_{1, \pi_1} \subset k_{2, \pi_2}$.
2. $\pi_1^{-f_{k_2/k_1}} N_{k_2/k_1}(\pi_2)$ is a root of unity if and only if there exists a finite extension $M/k_{2, \pi_2}$ such that $k_{1, \pi_1} \subset M$. If this is the case, we can take $M$ to be the degree $2\mu_{\infty}(k_1)$ subextension in $k_{2, \pi_2}$. Here, $\mu_{\infty}(k_1)$ is the set of roots of unity in $k_1$.

**Proof.** For $i = 1, 2$, let $k_{2, \pi_2}$ be the maximal unramified extension of $k_i$ and the maximal abelian extension of $k_i$, respectively. We recall that the Artin map $\Art_{k_i} : k_i^\times \to \text{Gal}(k_{ab}^{i}/k_i)$ associated with $k_i$ satisfies $\Art_{k_i}(\pi_i)|_{k_{1, \pi_1}} = 1$ and $\Art_{k_i}(\pi_i)|_{k_{2, \pi_2}} = \text{Frob}_{k_{2, \pi_2}}$, where $\text{Frob}_{k_{2, \pi_2}}$ is the geometric Frobenius of $k_i$.

1. Suppose $N_{k_2/k_1}(\pi_2) = \pi_1^{f_k}$. For any lift $\sigma \in G_{k_2}$ of $\Art_{k_2}(\pi_2)$, we have

$$\sigma|_{k_{1, \pi_1}} = (\Art_{k_2}(\pi_2)|_{k_{ab}^{i}})|_{k_{1, \pi_1}} = \Art_{k_1}(N_{k_2/k_1}(\pi_2))|_{k_{1, \pi_1}} = \Art_{k_1}(\pi_1)^{f_{k_2/k_1}}|_{k_{1, \pi_1}} = 1.$$

Since the intersection of the fixed fields (in $\mathbb{Q}_p$) of such $\sigma$'s is $k_{2, \pi_2}$, we obtain the desired result.

Conversely, suppose $k_{1, \pi_1} \subset k_{2, \pi_2}$. Then we have

$$\Art_{k_1}(N_{k_2/k_1}(\pi_2))|_{k_{1, \pi_1}} = \Art_{k_2}(\pi_2)|_{k_{1, \pi_1}} = (\Art_{k_2}(\pi_2)|_{k_{2, \pi_2}})|_{k_{1, \pi_1}} = 1$$

and

$$\Art_{k_1}(N_{k_2/k_1}(\pi_2))|_{k_{1, \pi_1}} = \Art_{k_2}(\pi_2)|_{k_{1, \pi_1}} = (\Art_{k_2}(\pi_2)|_{k_{2, \pi_2}})|_{k_{1, \pi_1}} = \text{Frob}_{k_{2, \pi_2}}|_{k_{1, \pi_1}} = \text{Frob}_{k_{1, \pi_1}}^{f_{k_2/k_1}}.$$ 

Thus we have $\Art_{k_1}(N_{k_2/k_1}(\pi_2)) = \Art_{k_1}(\pi_1^{f_k})$, which shows $N_{k_2/k_1}(\pi_2) = \pi_1^{f_k}$.

2. A very similar proof to that of (1) proceeds. Suppose that $\pi_1^{-f_{k_2/k_1}} N_{k_2/k_1}(\pi_2)$ is a root of unity. If we denote by $h$ the order of the set of roots of unity in $k_1$, then we have $N_{k_2/k_1}(\pi_2^h) = \pi_1^{fh}$. We see that any lift $\sigma \in G_{k_2}$ of $\Art_{k_2}(\pi_2^h)$ fixes $k_{1, \pi_1}$. This implies that $k_{1, \pi_1}$ is contained in a degree $h$ subextension in $k_{2, \pi_2}$.

Suppose that there exists a finite extension $M/k_{2, \pi_2}$ such that $k_{1, \pi_1} \subset M$. Then $M' := k_{1, \pi_1}k_{2, \pi_2}$ is a finite subextension in $k_{2, \pi_2}$. Put $h = [M' : k_{2, \pi_2}]$. Since $\Art_{k_2}(\pi_2^h)|_{M'}$ is the
identity map, we have \( \text{Art}_{k_1}(\text{Nr}_{k_2/k_1}(\pi_1^{\phi}))|_{k_1,\tau_1} = \text{id} \) and \( \text{Art}_{k_1}(\text{Nr}_{k_2/k_1}(\pi_2^{\phi}))|_{\tau^w} = \text{Frob}_{k_1}^{\phi} \). Thus we have \( \text{Art}_{k_1}(\text{Nr}_{k_2/k_1}(\pi_1^{\phi})) = \text{Art}_{k_1}(\pi_{1}^{\phi}) \), which shows \( \text{Nr}_{k_2/k_1}(\pi_2^{\phi}) = \pi_{1}^{\phi} \).

We recall that \( k_G \) is the Galois closure of \( k/Q_p \) and \( d_G := [k_G : Q_p] \).

**Lemma 2.8.** There exist a finite unramified extension \( k'/k_G \) and a uniformizer \( \pi' \) of \( k' \) which satisfy the following.

- \( \text{Nr}_{k'/k}((\pi')) = \pi_{k'/k} \),
- \( k_\pi \subset k'_{\pi'} \), where \( k'_{\pi'} \) is the Lubin-Tate extension of \( k' \) associated with \( \pi' \),
- the extension \( k'/Q_p \) is Galois, and
- \([k' : Q_p] = s d_G \) for some integer \( 1 \leq s \leq e_G \).

**Proof.** Let \( k_{G,0}/k \) be the maximal unramified subextension in \( k_G/k \). By [Sel, Chapter V, §6, Proposition 10], there exists an unramified extension \( k_0 \) over \( k_{G,0} \) of degree at most \([k_G : k_{G,0}] = e_G \) such that \( \pi = \text{Nr}_{k'/k_0}(\pi') \) for some \( \pi' \in (k')^\times \), where \( k' := k_G k_0 \). Since \( k_G/Q_p \) is Galois and \( k'/k \) is unramified, we see that \( k'/Q_p \) is Galois. We also see that \( \pi' \) is a uniformizer of \( k' \). Since \( k_G \cap k_0 = k_{G,0} \), we have \([k' : k_G] = [k_0 : k_{G,0}] \leq e_G \). Thus we obtain \([k' : Q_p] = [k' : k_G][k_G : Q_p] = s d_G \) for some integer \( 1 \leq s \leq e_G \). Furthermore, we have \( \text{Nr}_{k'/k}(\pi') = \text{Nr}_{k'/k_0}(\pi') = \text{Nr}_{k_0/k}(\pi) = \pi_{k'/k} \). By Lemma 2.7, we have \( k_\pi \subset k'_{\pi'} \).}

Now we are ready to prove Theorem 2.3.

**Proof of Theorem 2.3.** First we consider the case where \( k/Q_p \) is Galois. Replacing \( L \) by a finite extension, we may assume that \( L/K \) is Galois. Then \( V^{G_L} \) is a \( G_K \)-stable submodule of \( V \). By Lemma 2.6, there exists a finite extension \( K'/Kk \) such that any root \( \alpha \) of the characteristic polynomial of \( D^+_k(V^{G_L}) \) is of the form

\[
a = a^{tw'/w}, \quad a = \prod_{\tau \in \Gamma_k} \tau(\pi)^{-n_\tau},
\]

with some integers \( (n_\tau)_{\tau \in \Gamma_k} \) such that \( dh_1 \leq \sum_{\tau \in \Gamma_k} n_\tau \leq dh_2 \). Here, \( d := [k : Q_p] \). Put \( R := \sum_{\tau \in \Gamma_k} n_\tau \). Then we have

\[
\prod_{\sigma \in \Gamma_k} \sigma(a) = \prod_{\tau \in \Gamma_k} \prod_{\sigma \in \Gamma_k} \sigma(\pi)^{-n_\tau} = \prod_{\tau \in \Gamma_k} \text{Nr}_{k/Q_p}(\pi)^{-n_\tau} = \text{Nr}_{k/Q_p}(\pi)^{-R}.
\]

Since \( V^{G_L} \) has Weil weights in \( S \), we see that \( \sigma(a) \) is a \( q \)-Weil number of weight \( w \in S \) for any \( \sigma \in \Gamma_k \). Thus it follows from the condition \( w \neq 0 \) and the equation (2.6) that we have \( R \neq 0 \). Therefore, we obtain \( \text{Nr}_{k/Q_p}(\pi) \) is a \( q \)-Weil number of weight \( -w/h \) where \( h := R/d \in [h_1, h_2] \cap (1/d)Z \). This shows Theorem 2.3 (1). Now Theorem 2.3 (2) follows from the fact that we have \((q^w \text{Nr}_{k/Q_p}(\pi)^{-h})^d = \text{Nr}_{k/Q_p}(q^w \pi) \) and \((q^w \alpha)^{tw'/w} = q^{w'h} \alpha \) is a root of the characteristic polynomial of \( D^+_k(V^{G_{K'}}) \) (here, \( q_{K'} \) is the order of the residue field of \( K' \)). Thus we obtained a proof of Theorem 2.3 in the case where \( k/Q_p \) is Galois.

Next we consider the case where \( k/Q_p \) is not necessarily Galois. Take a finite extension \( k'/k_G \) and a uniformizer \( \pi' \) of \( k' \) as in Lemma 2.8. Put \( d' := [k' : Q_p] \). We have \( d' = s d_G \) for some \( 1 \leq s \leq e_G \). Let \( q' \) be the order of the residue field of \( k' \). Let \( L' \) be the composite field of \( L \) and \( k'_{\pi'} \), which is a finite extension of \( Kk'_{\pi'} \). Assume that \( V^{G_L} \) is not zero. Since \( V^{G_{K'}} \) is also not zero and the extension \( k'/Q_p \) is Galois, we know that \( \text{Nr}_{k'/Q_p}(\pi') \) is a \( q' \)-Weil number of weight \( -w/h \) for some \( w \in S \) and \( h \in [h_1, h_2] \cap (1/d')Z \). By the equation \( \text{Nr}_{k'/k}(\pi') = \pi_{k'/k} \), we have \( \text{Nr}_{k'/Q_p}(\pi') = (\text{Nr}_{k/Q_p}(\pi'))^{w/h} \), and hence \( \text{Nr}_{k/Q_p}(\pi) \) is a \( q \)-Weil number of weight \( -w/h \). Furthermore, we have \( q^w \text{Nr}_{k/Q_p}(\pi)^{-h} = (q^w \text{Nr}_{k/Q_p}(\pi)^{-h})^{tw'/w} \). This completes the proof of Theorem 2.3. \( \square \)
2.2 Proofs of Theorems 1.1 and 1.2

We prove Theorems 1.1 and 1.2 in the Introduction. We start with a proof of Theorem 1.2.

Proof of Theorem 1.2. Let the notation be as in the theorem. Replacing \( K \) by a finite extension, we may assume that \( X \) has good reduction over \( K \). Then we know that \( V \) is crystalline with Hodge-Tate weights in \([-i + r, r]\) (cf. [Fal], [Fa2]). We claim that \( V \) has Weil weight \( i - 2r \). Let \( K_0 \) be the maximal unramified subextension of \( K/\mathbb{Q}_p \). Put \( q_K = p^{i_k} \), the order of the residue field of \( K \). Let \( Y \) be the special fiber of a proper smooth model of \( X \) over the integer ring of \( K \). By the crystalline conjecture shown by Faltings [Fa2] (cf. [Ni], [Tsu]), we have an isomorphism \( D_{\mathbb{cris}}(H^i_{\mathbb{et}}(X_{\overline{K}}, \mathbb{Q}_p)) \simeq K_0 \otimes_{W(F_{q_K})} H^i_{\mathbb{cris}}(Y/W(F_{q_K})) \) of \( \varphi \)-modules over \( K_0 \). It follows from Corollary 1.3 of [CLS] (cf. [KM, Theorem 1] and [Na, Remark 2.2.4 (4)]) that the characteristic polynomial of \( K_0 \otimes_{W(F_{q_K})} H^i_{\mathbb{cris}}(Y/W(F_{q_K})) \) coincides with \( \text{char}_X(T) := \det(T - \text{Frob}_{q_K} | H^i_{\mathbb{cris}}(X_{\overline{K}}, \mathbb{Q}_p)) \) for any prime \( \ell \neq p \).

Hence we obtain the fact that the characteristic polynomial \( \text{char}_{V(-r)}(T) \) of \( D_{\mathbb{cris}}^\text{tor}(V(-r)) \) divides \( \text{char}_X(T) \). Thus it follows from the Weil Conjecture (cf. [De1], [De2]) that \( \text{char}_{V(-r)}(T) \) has algebraic integer coefficients and its roots are \( q_K \)-Weil numbers of weight \( i \). In particular, \( V \) has Weil weight \( i - 2r \) as desired. Now the result follows by Theorem 2.3. \( \square \)

Finally, we prove Theorem 1.1. Let \( A \) be an abelian variety over a \( p \)-adic field \( L \) and let \( \ell \) be any prime number. We denote by \( T_{\ell}(A) \) the \( \ell \)-adic Tate module of \( A \) and set \( V_{\ell}(A) := T_{\ell}(A \otimes_{\mathbb{Q}_p} \mathbb{Q}_\ell) \). It is well known that we have \( G_{\mathbb{K}} \)-equivariant isomorphisms \( V_{\ell}(A) \simeq H^2_{\text{et}}(A_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \) and \( V_{\ell}(A)/T_{\ell}(A) \simeq A(\overline{\mathbb{K}})[(\ell^\infty)] \). Here, \( A(\overline{\mathbb{K}})[(\ell^\infty)] \) is the \( \ell \)-power torsion subgroup of \( A(\overline{\mathbb{K}}) \). Furthermore, for an algebraic extension \( L/K \), the \( \ell \)-power torsion subgroup \( A(\mathbb{L})[(\ell^\infty)] \) of \( A(L) \) is finite if and only if \( V_{\ell}(A)^{G_L} = 0 \).

Below we denote by \( L \) any finite extension of \( k_{\mathbb{K}} \). Assume that \( A \) has potential good reduction and \( N_{L/k_{\mathbb{K}}} (\pi) \) satisfies the condition in the statement of Theorem 1.1. For the proof of Theorem 1.1, it is enough to show that both the \( p \)-part and the prime-to-\( p \) part of \( A(L)_{\text{tor}} \) are finite.

Finiteness of the \( p \)-part of \( A(L)_{\text{tor}} \): If we put \( W = V_p(A)^{G_L} \), then it is enough to show \( W = 0 \). Replacing \( L \) by a finite extension, we may suppose that the extension \( L/K \) is Galois. Then the \( G_{\mathbb{K}} \)-action on \( V_p(A) \) preserves \( W \), and thus the dual representation \( W^\vee \) of \( W \) is a quotient representation of \( H^2_{\text{et}}(A_{\overline{\mathbb{K}}}, \mathbb{Q}_p) \). By Theorem 1.2, we have \( W^\vee = (W^\vee)^{G_L} = 0 \), which implies \( W = 0 \) as desired.

Finiteness of the prime-to-\( p \) part of \( A(L)_{\text{tor}} \): The finiteness of the prime-to-\( p \) part of \( A(L)_{\text{tor}} \) follows from the following general property.

Proposition 2.9. Let \( A \) be an abelian variety over \( K \) with potential good reduction. Let \( M \) be an algebraic extension of \( K \) with finite residue field. Then the prime-to-\( p \) part of \( A(M)_{\text{tor}} \) is finite.

Proof. Replacing \( K \) and \( M \) by finite extensions, we may assume that \( A \) has good reduction over \( K \). It follows from the criterion of Néron-Ogg-Shafarevich [ST, Theorem 1] that the prime-to-\( p \) part of \( A(M)_{\text{tor}} \) has values in the maximal unramified subextension of \( M/K \), which is a finite extension of \( K \) by assumption on \( M \). Then the result follows from the main theorem of [Ma]. \( \square \)

Therefore, we obtained the proof of Theorem 1.1.

Remark 2.10. (This is pointed out by Yuichiro Taguchi.) We can construct an example which gives a negative answer to the question given in the Introduction for potential good reduction case. Let \( E \) be an elliptic curve over \( \mathbb{Q} \) with complex multiplication by the full ring of integers \( \mathcal{O}_F \) of an imaginary quadratic field \( F \). Let \( \psi = \psi_{E/F} \) be the Grössencharacter associated with \( E \). Let \( p \) be a prime number such that \( E \) has good ordinary reduction and \( p \) a prime ideal of \( \mathcal{O}_F \) above \( p \). If we set \( \pi := \psi(p) \), then \( \pi \) is a generator of \( p \) and we have \( p = \pi \overline{\pi} \). Here, \( \overline{\pi} \) is the complex conjugate of \( \pi \). Note that \( \pi \) is a \( p \)-Weil number of weight 1. Let \( K = k \) be the completion of \( F \) at \( p \). By definition, we have \( K = k = Q_p \), and \( \pi \) is a uniformizer of them. If we identify a decomposition group of \( G_F \) at \( p \) with \( G_K \), then the action of \( G_K \) on the set of \( p \)-power torsion points of \( E(\overline{K}) \) is
given by the Lubin-Tate character $\chi_{\pi}$ associated with $\pi$. In particular, we see that $E(K_{k_\pi})[p^\infty]$ is infinite.

References


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