2025 年度　解析 III 演習 (担当：松澤　寛)　レジメ

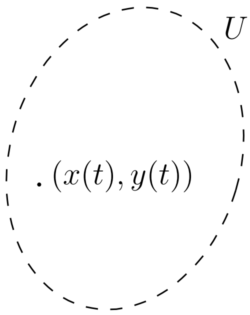
No.4　合成関数の微分

まとめ 1（合成関数）

- 関数 $z = f(x, y)$ は点 (a, b) の近傍 U で定義されているとする.
- 関数 $x(t)$, $y(t)$ はある開区間 I で定義されているとする.
- 点 $(x(t), y(t))$ が

$$(x(t), y(t)) \in U \quad ({}^{\forall} t \in I)$$

であれば合成関数 $z = z(t) = f(x(t), y(t))$ $(t \in I)$ が定義される.



まとめ 2（合成関数の微分）

上の状況下でさらに $z = f(x, y)$ は点 (a, b) で全微分可能，$x = x(t)$, $y = y(t)$ は $t = c$ で微分可能で $x(c) = a$, $y(c) = b$ ならば $z = z(t) = f(x(t), y(t))$ は $t = c$ で微分可能で

$$\boldsymbol{\frac{dz}{dt}(c) = \frac{\partial z}{\partial x}(a, b)\frac{dx}{dt}(c) + \frac{\partial z}{\partial y}(a, b)\frac{dy}{dt}(c)}$$
$$[\boldsymbol{z'(c) = f_x(a, b)x'(c) + f_y(a, b)y'(c)}]$$

導関数の場合

$z = f(x, y)$ が U の各点で全微分可能，$x = x(t)$, $y = y(t)$ がすべての $t \in I$ で微分可能であるとき，$z = z(t) = f(x(t), y(t))$ はすべての $t \in I$ で微分可能で

$$\boldsymbol{\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)\end{aligned}}$$

まとめ 3（合成関数の（偏）微分法）

- $z = f(x, y)$ は点 (a, b) で全微分可能
- $x = x(u, v)$, $y = y(u, v)$ は点 (c, d) で全（偏）微分可能で $a = x(c, d)$, $b = y(c, d)$

であるならば $z = z(u, v) = f(x(u, v), y(u, v))$ は点 (c, d) で全（偏）微分可能で

$$\boldsymbol{\frac{\partial z}{\partial u}(c,d) = \frac{\partial z}{\partial x}(a,b)\frac{\partial x}{\partial u}(c,d) + \frac{\partial z}{\partial y}(a,b)\frac{\partial y}{\partial u}(c,d)}$$
$$[\boldsymbol{z_u(c,d) = f_x(a,b)x_u(c,d) + f_y(a,b)y_u(c,d)}]$$

$$\boldsymbol{\frac{\partial z}{\partial v}(c,d) = \frac{\partial z}{\partial x}(a,b)\frac{\partial x}{\partial v}(c,d) + \frac{\partial z}{\partial y}(a,b)\frac{\partial y}{\partial v}(c,d)}$$
$$[\boldsymbol{z_v(c,d) = f_x(a,b)x_v(c,d) + f_y(a,b)y_v(c,d)}]$$

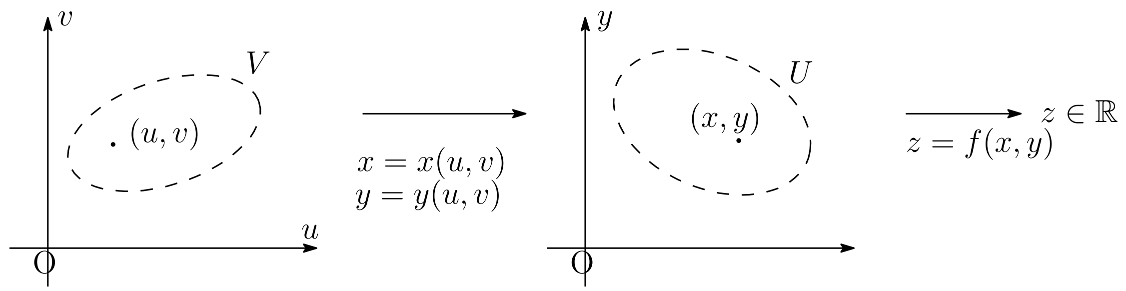
偏導関数の場合

- $z = f(x, y)$ は U の各点で全微分可能
- $x = x(u, v)$, $y = y(u, v)$ は uv 平面の領域 V の各点で全微分可能で，$(x(u, v), y(u, v)) \in U$ $({}^\forall(u, v) \in V)$

とするとき，$z = z(u, v) = f(x(u, v), y(u, v))$ は V の各点 (u, v) で全微分可能で

$$\boldsymbol{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$

$$\boldsymbol{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$



演習 1

$z=\log\dfrac{x}{y}$, $x=e^t+e^{-t}$, $y=e^t-e^{-t}$ のとき $\dfrac{dz}{dt}$ を求めよ.

解

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

ここで $z=\log x-\log y$ より

$$\frac{\partial z}{\partial x}=\frac{1}{x},\ \frac{\partial z}{\partial y}=-\frac{1}{y}$$

また

$$\frac{dx}{dt}=e^t-e^{-t},\quad \frac{dy}{dt}=e^t+e^{-t}$$

よって

$$\begin{aligned}\frac{dz}{dt}&=\frac{1}{x}(e^t-e^{-t})-\frac{1}{y}(e^t+e^{-t})\\&=\frac{1}{e^t+e^{-t})}(e^t-e^{-t})-\frac{1}{e^t-e^{-t}}(e^t+e^{-t})\\&=\frac{(e^t-e^{-t})^2-(e^t+e^{-t})^2}{(e^t+e^{-t})(e^t-e^{-t})}\\&=-\frac{4}{e^{2t}-e^{-2t}}\quad //\end{aligned}$$

演習 2（講義でも扱うかもしれませんが大切な問題なので）

$z=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$ の時 $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial \theta}$, $\dfrac{\partial^2 z}{\partial r^2}$, $\dfrac{\partial^2 z}{\partial \theta^2}$ を求めよ．ただし，$f(x,y)$ は第 2 次偏導関数まで連続とする．

解 $z=f(r\cos\theta, r\sin\theta)$

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta)$$

ここで $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ はそれぞれ $f_x(r\cos\theta, r\sin\theta)$, $f_y(r\cos\theta, r\sin\theta)$ であり，それぞれ何かある x, y の 2 変数関数の x, y に $r\cos\theta$, $r\sin\theta$ が代入されていることに注意.

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2}&=\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\cos\theta+\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\sin\theta \qquad \leftarrow \cos\theta, \sin\theta \text{ は定数扱い}\\
&=\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial r}\right)\cos\theta+\left(\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right)\sin\theta\\
&=\left(\frac{\partial^2 z}{\partial x^2}\cos\theta+\frac{\partial^2 z}{\partial y\partial x}\sin\theta\right)\cos\theta+\left(\frac{\partial^2 z}{\partial x\partial y}\cos\theta+\frac{\partial^2 z}{\partial y^2}\sin\theta\right)\sin\theta \qquad \leftarrow \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}\\
&=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+2\frac{\partial^2 z}{\partial y\partial x}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta
\end{aligned}$$

次に $\dfrac{\partial^2 z}{\partial \theta^2}$ を計算する．

$$\left[\ \frac{\partial z}{\partial \theta}=\underset{f_x(r\cos\theta,\, r\sin\theta)}{\frac{\partial z}{\partial x}}(-r\sin\theta)+\underset{f_y(r\cos\theta,\, r\sin\theta)}{\frac{\partial z}{\partial y}}(r\cos\theta) \qquad \leftarrow \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \text{ にも } \theta \text{ があるので積の微分}\ \right]$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial \theta^2}&=\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right)(-r\sin\theta)+\frac{\partial z}{\partial x}\frac{\partial}{\partial \theta}(-r\sin\theta)+\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right)(r\cos\theta)+\frac{\partial z}{\partial y}\frac{\partial}{\partial \theta}(r\cos\theta)\\
&=\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial \theta}+\frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial \theta}\right)(-r\sin\theta)+\frac{\partial z}{\partial x}(-r\cos\theta)\\
&\qquad+\left(\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial \theta}+\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial \theta}\right)+\frac{\partial z}{\partial y}(-r\sin\theta)\\
&=\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta)+\frac{\partial^2 z}{\partial y\partial x}(r\cos\theta)\right)(-r\sin\theta)+\frac{\partial z}{\partial x}(-r\cos\theta)\\
&\qquad+\left(\frac{\partial^2 z}{\partial x\partial y}(-r\sin\theta)+\frac{\partial^2 z}{\partial y^2}(r\cos\theta)\right)+\frac{\partial z}{\partial y}(-r\sin\theta) \qquad \leftarrow \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}\\
&=r^2\frac{\partial^2 z}{\partial x^2}\sin^2\theta-2r^2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta+r^2\frac{\partial^2 z}{\partial y^2}\cos^2\theta-r\frac{\partial z}{\partial x}\cos\theta-r\frac{\partial z}{\partial y}\sin\theta \qquad //
\end{aligned}$$