Torsion of abelian varieties and Lubin-Tate extensions

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Abstract

We show that, for an abelian variety defined over a p-adic field K which has potential good reduction, its torsion subgroup with values in the composite field of K and a certain Lubin-Tate extension over a p-adic field is finite.

1 Introduction

Let p be a prime number and A an abelian variety over a p-adic field K (here, a *p-adic field* is a finite extension of $\mathbb{Q}_p$). For an algebraic extension L/K, we denote by $A(L)$ the group of L-rational points of A and also denote by $A(L)_{\mathrm{tor}}$ its torsion subgroup. We are interested in determining whether $A(L)_{\mathrm{tor}}$ is finite or not. The most basic result is given by Mattuck [Ma]; $A(L)_{\mathrm{tor}}$ is finite if L is a finite extension of K. Thus our main interest is the case where L is an infinite algebraic extension of K. For this, Imai's result [Im] is well known. He showed that $A(K(\mu_{p^\infty}))_{\mathrm{tor}}$ is finite if A has potential good reduction, where μ_{p^∞} denotes the group of p-power roots of unity in a fixed separable closure $\overline{K}$ of K. Since the field $K(\mu_{p^\infty})$ is the composite field of K and the Lubin-Tate extension over $\mathbb{Q}_p$ associated with a uniformizer p of $\mathbb{Q}_p$, we naturally have the following question.

Question. Let A be an abelian variety over a p-adic field K. Let k_π be the Lubin-Tate extension associated with a uniformizer π of a p-adic field k. Then, is $A(Kk_\pi)_{\mathrm{tor}}$ finite?

In the case of Imai's theorem ($k = \mathbb{Q}_p$ and $\pi = p$), the answer of the question is affirmative for potential good reduction cases, that is, the case where A has potential good reduction. However, the question sometimes has a negative answer. For example, if A is a Tate curve over K, $k = \mathbb{Q}_p$ and $\pi = p$, then $A(Kk_\pi)[p^\infty] = A(K(\mu_{p^\infty}))[p^\infty]$ is clearly infinite. We also have an example even for potential good reduction cases as given in Remark 2.10.

The aim of this paper is to give a sufficient condition on k and π so that the question has an affirmative answer for potential good reduction cases. Let k, π and k_π be as above. Let q be the order of the residue field of k. We denote by k_G the Galois closure of $k/\mathbb{Q}_p$. We put $d_G = [k_G : \mathbb{Q}_p]$ and denote by e_G the ramification index of the extension k_G/k. We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Our main result is as follows (see Definitions 2.1 and 2.2 for some undefined notion).

Theorem 1.1. *Let A be an abelian variety over a p-adic field K with potential good reduction. If $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$ is not a q-Weil integer of weight sd_G/t for any integers $1 \leq s \leq e_G$ and $1 \leq t \leq sd_G$, then $A(Kk_\pi)_{\mathrm{tor}}$ is finite.*

Applying Theorem 1.1 to the case where $k = \mathbb{Q}_p$ and $\pi = p$, we can recover Imai's theorem. We should note that there is another generalization of Imai's theorem which is given by Kubo and Taguchi [KT]. The main result of *loc. cit.* states that the torsion subgroup of $A(K(K^{1/p^\infty}))$ is

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finite, where A is an abelian variety over K with potential good reduction and $K(K^{1/p^\infty})$ is the extension field of K by adjoining all p-power roots of all elements of K.

For the proof of the above theorem, the essential difficulty appears in the finiteness of the p-power torsion part $A(Kk_\pi)[p^\infty]$ of $A(Kk_\pi)_{\mathrm{tors}}$. For this, we proceed our arguments in more general settings. We study not only abelian varieties but also (general) proper smooth varieties.

Theorem 1.2. *Let X be a proper smooth variety over a p-adic field K with potential good reduction. Let V be a $\mathrm{Gal}(\overline{K}/K)$-stable subquotient of $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))$ with $i \neq 2r$. Assume that $V^{\mathrm{Gal}(\overline{K}/L)} \neq 0$ for some finite extension L/Kk_π. Then $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$ is a q-Weil number of weight $-(i-2r)/h$ for some non-zero $h \in [-i+r, r] \bigcap \left(\bigcup_{s\in\mathbb{Z}, 1\le s\le e_G}(1/sd_G)\mathbb{Z}\right)$. Moreover, $q^r\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{-h}$ is an algebraic integer.*

Applying Theorem 1.2 to the case where $k = \mathbb{Q}_p$, $\pi = p$ and i is odd, we obtain [CSW, Corollary 1.6]. (Note that *loc. cit.* studies the vanishing of not only $H^0(\mathrm{Gal}(\overline{K}/L), V)$ (as our result) but also $H^j(\mathrm{Gal}(\overline{K}/L), V)$ for all j.) The assumption $i \neq 2r$ in Theorem 1.2 is essential as explained in the Introduction of [KT]. The key ingredients for our proof are the theory of locally algebraic representations (cf. [Se2]) and some "weight arguments" of eigenvalues of Frobenius on various objects. For weight arguments, we use p-adic Hodge theory related with Lubin-Tate characters and results on weights of a Frobenius operator on crystalline cohomologies (cf. [CLS], [KM], [Na]).

We hope our results can be useful for future studies in Iwasawa theory, for example, control theorems of Selmer groups for abelian varieties over certain p-adic extensions of number fields. In fact, arguments of [KT, Section 6] seem to be familiar with our results.

Notation : In this paper, we fix algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$, respectively, and we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. If F is a p-adic field, we denote by G_F and U_F the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ of F and the unit group of the integer ring of F, respectively. We also denote by F^{ur} and I_F the maximal unramified extension of F in $\overline{\mathbb{Q}}_p$ and the inertia subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}})$ of G_F, respectively. We set $\Gamma_F := \mathrm{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)$. If F'/F is a finite extension, we denote by $f_{F'/F}$ the residual extension degree of F'/F, that is, the extension degree of the residue fields corresponding to F'/F. We put $f_F = f_{F/\mathbb{Q}_p}$. Finally, any p-adic representation of G_F in this paper is of finite dimension.

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2 Proofs of main theorems

Our goal is to prove results in the Introduction. We often use p-adic Hodge theory. For the basic notion of this theory, it is helpful for the reader to refer [Fo1] and [Fo2]. In this paper, we normalize the Hodge-Tate weight so that the Hodge-Tate weight of $\mathbb{Q}_p(1)$ is one.

Definition 2.1. Let $q_0 > 1$ be an integer. A q_0-*Weil number* (*resp.* q_0-*Weil integer*) *of weight* w is an algebraic number (resp. algebraic integer) α such that $|\iota(\alpha)| = q_0^{w/2}$ for all embeddings $\iota\colon \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.

Definition 2.2. Let F be a p-adic field with residual extension degree $f = f_F$ and $F_0/\mathbb{Q}_p$ the maximal unramified subextension of $F/\mathbb{Q}_p$. We denote by $\varphi_{F_0}\colon F_0 \to F_0$ the arithmetic Frobenius of F_0, that is, the (unique) lift of p-th power map on the residue field of F_0.
(1) Let D be a φ-module over F_0, that is, a finite dimensional F_0-vector space with φ_{F_0}-semilinear map $\varphi\colon D \to D$. Then $\varphi^f\colon D \to D$ is a F_0-linear map. We call $\det(T - \varphi^f \mid D)$ the *characteristic polynomial of D*.

(2) For a $\mathbb{Q}_p$-representation U of G_F, we set $D^F_{\mathrm{cris}}(U) := (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} U)^{G_F}$ and $D^F_{\mathrm{st}}(U) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} U)^{G_F}$, which are filtered φ-modules over F. Here, B_{cris} and B_{st} are usual p-adic period rings. Note that we have $D^F_{\mathrm{cris}}(U) = D^F_{\mathrm{st}}(U)$ if U is crystalline.
(3) Let S be a set of rational numbers. Let U be a potentially semi-stable $\mathbb{Q}_p$-representation of G_F. Suppose that $U|_{G_{F'}}$ is semi-stable for a finite extension F' of F with residue field $\mathbb{F}_{q'}$. We say that U has *Weil weights in* S if any root of the characteristic polynomial of $D^{F'}_{\mathrm{st}}(U)$ is a q'-Weil number of weight w for some $w \in S$. (Note that this definition does not depend on the choice of F'.)

Let K and k be finite extensions of $\mathbb{Q}_p$. Let q be the order of the residue field of k, π a uniformizer of k and k_π the Lubin-Tate extension of k associated with π. The following theorem is a key to the proof of our main results.

Theorem 2.3. *Let S be a subset of $\mathbb{Q} \smallsetminus \{0\}$. Let V be a semi-stable $\mathbb{Q}_p$-representation of G_K with Hodge-Tate weights in $[h_1, h_2]$. Assume that V has Weil weights in S and $V^{\mathrm{Gal}(\overline{K}/L)} \neq 0$ for some finite extension L/Kk_π. Then*
(1) $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$ *is a q-Weil number of weight $-w/h$ for some $w \in S$ and some non-zero $h \in [h_1, h_2] \bigcap \left(\bigcup_{s\in\mathbb{Z}, 1\leq s\leq e_G} (1/sd_G)\mathbb{Z}\right)$.*
(2) *If the coefficients of the characteristic polynomial of $D^K_{\mathrm{st}}(V(-r))$ are algebraic integers for some integer r, then we can choose h in* (1) *so that $q^r \mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{-h}$ is an algebraic integer.*

2.1 Proof of Theorem 2.3

In this section, we prove Theorem 2.3. We begin with some lemmas.

Lemma 2.4. *Let $(n_\sigma)_{\sigma\in\Gamma_K}$ be a family of integers. If there exists an open subgroup U of U_K with the property that $\prod_{\sigma\in\Gamma_K} \sigma(x)^{n_\sigma} = 1$ for any $x \in U$, then we have $n_\sigma = 0$ for any $\sigma \in \Gamma_K$.*

Proof. Replacing U by a finite index subgroup, we may assume that the p-adic logarithm map is defined on U. Then we have $\sum_{\sigma\in\Gamma_K} n_\sigma \sigma(\log x) = 0$ for any $x \in U$ by assumption. Since $\log U$ is an open ideal of the ring of integers of K, we obtain $\sum_{\sigma\in\Gamma_K} n_\sigma \sigma(y) = 0$ for any $y \in K$. Although the desired fact $n_\sigma = 0$ for any $\sigma \in \Gamma_K$ follows from Dedekind's theorem [Bo, §6, no. 2, Corollaire 2] immediately, we also give a direct proof for this. Take any $\alpha \in K$ such that $K = \mathbb{Q}_p(\alpha)$ and let $\Gamma_K = \{\sigma_1 = \mathrm{id}, \sigma_2, \dots, \sigma_c\}$ where $c := [K : \mathbb{Q}_p]$. Then we have $(n_{\sigma_1}, n_{\sigma_2}, \dots, n_{\sigma_c})X = \mathbf{0}$ where X is the $c \times c$ matrix with (i, j)-th component $\sigma_i(\alpha)^{j-1}$. Since $\det X = \prod_{j>i}(\sigma_j(\alpha) - \sigma_i(\alpha)) \neq 0$, we obtain $n_{\sigma_1} = n_{\sigma_2} = \cdots = n_{\sigma_c} = 0$. $\square$

We denote by $\chi_\pi \colon G_k \to k^\times$ the Lubin-Tate character associated with π. If we regard χ_π as a continuous character $k^\times \to k^\times$ by the local Artin map with arithmetic normalization, then χ_π is characterized by the property that $\chi_\pi(\pi) = 1$ and $\chi_\pi(u) = u^{-1}$ for any $u \in U_k$.

Lemma 2.5. *Let E be a p-adic field and V an E-representation of G_K. Assume that $k/\mathbb{Q}_p$ is Galois, V is Hodge-Tate and the G_{Kk_π}-action on V factors through a finite quotient. Then, there exist finite extensions K'/K and E'/E with $K', E' \supset k$ such that any Jordan-Hölder factor of $(V \otimes_E E')|_{G_{K'}}$ is of the form $E'(\prod_{\sigma\in\Gamma_k} \sigma^{-1} \circ \chi_\pi^{r_\sigma})$ for some $r_\sigma \in \mathbb{Z}$. Moreover, r_σ is a Hodge-Tate weight of V.*

Proof. Replacing K by a finite extension, we may assume that G_{Kk_π} acts trivially on V and K is a finite Galois extension of k. Since the G_K-action on V factors through the abelian group $\mathrm{Gal}(Kk_\pi/K)$, it follows from Schur's lemma that, for a finite extension E'/E of sufficiently large degree, any Jordan-Hölder factor W of $V \otimes_E E'$ is of dimension 1. Our goal is to show that W is of the required form. We may assume $E' = E \supset K$.

Let $\rho \colon G_K \to GL_E(W) \simeq E^\times$ be the continuous homomorphism given by the G_K-action on W. Let $\tilde{E}$ be the Galois closure of $E/\mathbb{Q}_p$ and take any finite extension K''/K which contains $\tilde{E}$. Since W is Hodge-Tate, it follows from [Se2, Chapter III, A.5, Theorem 2] that there exists an open subgroup I of $I_{K''}$ such that $\rho = \prod_{\sigma\in\Gamma_E} \sigma^{-1}\circ\chi_{\sigma E}^{n_\sigma}$ on I for some integer n_σ. Here, $\chi_{\sigma E}\colon G_{\sigma E}\to U_{\sigma E}$ is the Lubin-Tate character associated with σE (it depends on the choice of a uniformizer of σE, but its restriction to the inertia subgroup does not). Put $\tilde\rho = \prod_{\sigma\in\Gamma_E} \sigma^{-1}\circ\chi_{\sigma E}^{n_\sigma}$, considered as a character of $G_{K''}$. Replacing K'' by a finite extension, we may assume the following:

- $K''/\mathbb{Q}_p$ is Galois, $\mathrm{Gal}(k_\pi/(k_\pi\cap K''))$ is torsion free and $\rho=\tilde\rho$ on $I_{K''}$.

Since $\rho|_{G_{Kk_\pi}}$ is trivial, we have that $\tilde\rho$ is trivial on $I_{K''}\cap G_{Kk_\pi} = G_{(K'')^{\mathrm{ur}}k_\pi}$. Hence, putting $N' = \mathrm{Gal}((K'')^{\mathrm{ur}}k_\pi/(K'')^{\mathrm{ur}})$, we may regard $\tilde\rho|_{I_{K''}}$ as a representation of N'. Put $N=\mathrm{Gal}(k^{\mathrm{ur}}k_\pi/k^{\mathrm{ur}})$. Then N' is canonically isomorphic to a torsion free finite index subgroup of $N\simeq U_k$, and thus we regard N' as a subgroup of N.

Now we claim that $\tilde\rho|_{I_{K''}}$, regarded as a continuous character $N'\to\tilde E^\times$, extends to a continuous character $\hat\rho\colon N\to\overline{\mathbb{Q}}_p^\times$. It follows from the theory of elementary divisors that we may regard $N = N_{\mathrm{tor}}\oplus(\oplus_{i=1}^d\mathbb{Z}_p)\supset\{0\}\oplus(\oplus_{i=1}^d p^{m_i}\mathbb{Z}_p) = N'$ with some integer $m_i\ge 0$. Here, N_{tor} is the torsion subgroup of N and $d:=[k:\mathbb{Q}_p]$. Hence it suffices to show that any continuous character $p^m\mathbb{Z}_p\to\overline{\mathbb{Q}}_p^\times$ with $m>0$ extends to $\mathbb{Z}_p\to\overline{\mathbb{Q}}_p^\times$, but this is clear.

By local class field theory, we may regard $\tilde\rho|_{I_{K''}}$ and $\hat\rho$ as characters of $U_{K''}$ and U_k, respectively. It follows from the construction of $\hat\rho$ that we have $\tilde\rho(x)=\hat\rho(\mathrm{Nr}_{K''/k}(x))$ for $x\in U_{K''}$. In particular, we have

$$\tilde\rho(x)=\tilde\rho(\tau x) \tag{2.1}$$

for $x\in U_{K''}$ and $\tau\in\mathrm{Gal}(K''/k)$. On the other hand, by definition of $\tilde\rho$ and the condition that $K''/\mathbb{Q}_p$ is Galois, we have

$$\tilde\rho(x)=\prod_{\sigma\in\Gamma_E}\sigma^{-1}\mathrm{Nr}_{K''/\sigma E}(x^{-1})^{n_\sigma} = \prod_{\tilde\sigma\in\Gamma_{K''}}\tilde\sigma^{-1}(x^{-1})^{n_{\tilde\sigma}} \tag{2.2}$$

for $x\in U_{K''}$ where $n_{\tilde\sigma}:=n_\sigma$ if $\tilde\sigma|_E=\sigma$.

We claim that $n_{\tilde\sigma}=n_{\tilde\sigma'}$ if $\tilde\sigma|_k=\tilde\sigma'|_k$. By (2.1) and (2.2), we have

$$\prod_{\tilde\sigma\in\Gamma_{K''}}\tilde\sigma^{-1}(x^{-1})^{n_{\tau\tilde\sigma}} = \prod_{\tilde\sigma\in\Gamma_{K''}}\tilde\sigma^{-1}(x^{-1})^{n_{\tilde\sigma}} \tag{2.3}$$

for $x\in U_{K''}$ and $\tau\in\mathrm{Gal}(K''/k)$. Choosing a lift $\hat\sigma\in\Gamma_{K''}$ for each element of $\mathrm{Gal}(k/\mathbb{Q}_p)$, we have a decomposition $\Gamma_{K''}=\bigcup_{\hat\sigma}\hat\sigma\mathrm{Gal}(K''/k)$. Since $k/\mathbb{Q}_p$ is Galois, we see that $\mathrm{Gal}(K''/k)$ acts on $\hat\sigma\mathrm{Gal}(K''/k)$ stably and this action is transitive. By Lemma 2.4, we know that the family $(n_{\tilde\sigma})_{\tilde\sigma\in\Gamma_{K''}}$ is determined uniquely by the restriction of $\prod_{\sigma\in\Gamma_{K''}}(\tilde\sigma^{-1})^{n_{\tilde\sigma}}$ to any open subgroup of $U_{K''}$. Hence the equation (2.3) gives $n_{\tilde\sigma}=n_{\tilde\sigma'}$ if $\tilde\sigma|_k=\tilde\sigma'|_k$ as desired.

For any $\sigma\in\Gamma_k$, we define $r_\sigma:=n_{\tilde\sigma}$ for a lift $\tilde\sigma\in\Gamma_{K''}$ of σ, which is independent of the choice of $\tilde\sigma$ by the claim just above. Then we see $\tilde\rho(x)=\prod_{\tilde\sigma\in\Gamma_{K''}}\tilde\sigma^{-1}(x^{-1})^{n_{\tilde\sigma}} = \prod_{\sigma\in\Gamma_k}\sigma^{-1}\mathrm{Nr}_{K''/k}(x^{-1})^{r_\sigma}$ for $x\in U_{K''}$. This implies

$$\tilde\rho=\prod_{\sigma\in\Gamma_k}\sigma^{-1}\circ\chi_\pi^{r_\sigma}$$

on $I_{K''}$. Now we define $\psi\colon G_K\to E^\times$ by $\psi:=\rho\cdot\left(\prod_{\sigma\in\Gamma_k}\sigma^{-1}\circ\chi_\pi^{r_\sigma}\right)^{-1}$. Then ψ is trivial on $I_{K''}$ since $\rho=\tilde\rho=\prod_{\sigma\in\Gamma_k}\sigma^{-1}\circ\chi_\pi^{r_\sigma}$ on $I_{K''}$. Furthermore, ψ is trivial on G_{Kk_π} since χ_π and ρ are trivial on G_{Kk_π}. Therefore, putting $K'=(K'')^{\mathrm{ur}}\cap Kk_\pi$, then K'/K is a finite extension and ψ is trivial on $G_{K'}$.

Finally, we note that r_σ is a Hodge-Tate weight of V by [Se2, Chapter III, A.5, Theorem 2]. This is the end of the proof. □

Lemma 2.6. *Let E be a p-adic field and V an E-representation of G_K. Assume that $k/\mathbb{Q}_p$ is Galois, V is potentially semi-stable with Hodge-Tate weights in $[h_1,h_2]$ and the G_{Kk_π}-action on V factors through a finite quotient. Then, there exists a finite extension K'/Kk which satisfies the* *following property: $V|_{G_{K'}}$ is semi-stable and, for any root α of the characteristic polynomial of $D^{K'}_{\mathrm{st}}(V)$, we have*

$$\alpha = a^{f_{K'/k}}, \quad a = \prod_{\tau\in\Gamma_k} \tau(\pi)^{-n_\tau}$$

for some integers $(n_\tau)_{\tau\in\Gamma_k}$ such that $dh_1 \le \sum_{\tau\in\Gamma_k} n_\tau \le dh_2$. Here, $d := [k : \mathbb{Q}_p]$.

Proof. By Lemma 2.5, there exist finite extensions K'/K and E'/E with $E', K' \supset k$ which satisfy the following:

– $V|_{G_{K'}}$ is semi-stable and any Jordan-Hölder factor W of $(V \otimes_E E')|_{G_{K'}}$ is of the form $E'(\prod_{\sigma\in\Gamma_k} \sigma^{-1} \circ \chi_\pi^{r_\sigma})$ for some $r_\sigma \in [h_1, h_2]$. In particular, W is crystalline.

Replacing E by a finite extension, we may assume $E' = E$. Now we take a root α of the characteristic polynomial of $D^{K'}_{\mathrm{st}}(V)$, and choose W so that α is a root of the the characteristic polynomial of $D^{K'}_{\mathrm{cris}}(W)$.

To study α, we first consider the characteristic polynomial of $D^{K'}_{\mathrm{cris}}(E(\sigma^{-1} \circ \chi_\pi^{r_\sigma}))$ for $\sigma \in \Gamma_k$. We note that we have an isomorphism $k(\sigma^{-1} \circ \chi_\pi^{r_\sigma})^{\mathrm{ss}} \simeq k(\chi_\pi^{r_\sigma})^{\mathrm{ss}}$ of $\mathbb{Q}_p[G_{K'}]$-modules (here, "ss" stands for the semi-simplification of $\mathbb{Q}_p[G_{K'}]$-modules). In fact, for any $g \in G_{K'}$, we have

$$\begin{aligned}\mathrm{Tr}_{\mathbb{Q}_p}(g \mid k(\sigma^{-1} \circ \chi_\pi^{r_\sigma})) &= \mathrm{Tr}_{k/\mathbb{Q}_p}(\mathrm{Tr}_k(g \mid k(\sigma^{-1} \circ \chi_\pi^{r_\sigma})) = \mathrm{Tr}_{k/\mathbb{Q}_p}(\sigma^{-1}\chi_\pi^{r_\sigma}(g))\\ &= \mathrm{Tr}_{k/\mathbb{Q}_p}(\chi_\pi^{r_\sigma}(g)) = \mathrm{Tr}_{k/\mathbb{Q}_p}(\mathrm{Tr}_k(g \mid k(\chi_\pi^{r_\sigma})) = \mathrm{Tr}_{\mathbb{Q}_p}(g \mid k(\chi_\pi^{r_\sigma})).\end{aligned}$$

(Here, for a representation U of a group G over a field F and $g \in G$, we denote by $\mathrm{Tr}_F(g \mid U)$ the trace of the g-action on the F-vector space U.) Therefore, we have

$$\det(T - \varphi^{f_{K'}} \mid D^{K'}_{\mathrm{cris}}(E(\sigma^{-1} \circ \chi_\pi^{r_\sigma}))) = \det(T - \varphi^{f_{K'}} \mid D^{K'}_{\mathrm{cris}}(k(\chi_\pi^{r_\sigma})))^{[E:k]}. \tag{2.4}$$

To study the roots of (2.4), we recall the explicit description of $D^k_{\mathrm{cris}}(k(\chi_\pi^{-1}))$ (cf. [Con, Proposition B.4]. See also [Col, Proposition 9.10]). Let k_0 be the maximal unramified subextension of $k/\mathbb{Q}_p$. By definition, we have $f_k = [k_0 : \mathbb{Q}_p]$ and $q = p^{f_k}$. Then $D^k_{\mathrm{cris}}(k(\chi_\pi^{-1}))$ is a free $(k_0 \otimes_{\mathbb{Q}_p} k)$-module of rank one, and we can take a basis $\mathbf{e}$ of $D^k_{\mathrm{cris}}(k(\chi_\pi^{-1}))$ such that $\varphi^{f_k}(\mathbf{e}) = (1 \otimes \pi)\mathbf{e}$. We claim

$$\det(T - \varphi^{f_k} \mid D^k_{\mathrm{cris}}(k(\chi_\pi^{-1}))) = \prod_{0\le i\le f_k-1} E^{\varphi^i}(T) \tag{2.5}$$

where $E(T) = T^e + \sum_{j=0}^{e-1} a_j T^j \in k_0[T]$ is the minimal polynomial of π over k_0 and $E^{\varphi^i}(T) = T^e + \sum_{j=0}^{e-1} \varphi^i(a_j)T^j$. To show this, it suffices to show that the characteristic polynomial of the homomorphism $1 \otimes \pi\colon k_0 \otimes_{\mathbb{Q}_p} k \to k_0 \otimes_{\mathbb{Q}_p} k$ of k_0-modules coincides with the right hand side of (2.5). (Here, the k_0-action on $k_0 \otimes_{\mathbb{Q}_p} k$ is given by $a.(x \otimes y) := ax \otimes y$ for $a, x \in k_0$ and $y \in k$.) We consider a natural isomorphism

$$k_0 \otimes_{\mathbb{Q}_p} k_0 \simeq \oplus_{j\in\mathbb{Z}/f_k\mathbb{Z}} k_{0,j},\ a \otimes b \mapsto (a\varphi^j(b))_j$$

where $k_{0,j} = k_0$ for all j. For $0 \le s \le f_k - 1$, let $e_s \in k_0 \otimes_{\mathbb{Q}_p} k_0$ be the element which corresponds to $(\delta_{sj})_j \in \oplus_{j\in\mathbb{Z}/f_k\mathbb{Z}} k_{0,j}$ where δ_{sj} is the Kronecker delta. Then $\{e_j(1 \otimes \pi^i) \mid 0 \le j \le f_k - 1,\ 0 \le i \le e-1\}$ is a k_0-basis of $k_0 \otimes_{\mathbb{Q}_p} k$. We see that the matrix of $1 \otimes \pi\colon k_0 \otimes_{\mathbb{Q}_p} k \to k_0 \otimes_{\mathbb{Q}_p} k$ associated with the ordered basis $\langle e_0, \dots, e_{f_k-1}, e_0(1 \otimes \pi), \dots, e_{f_k-1}(1 \otimes \pi), \dots, e_0(1 \otimes \pi^{e-1}), \dots, e_{f_k-1}(1 \otimes \pi^{e-1})\rangle$ is

$$\begin{pmatrix} O & O & \cdots & -A_0 \\ I_{f_k} & O & \cdots & -A_1 \\ \vdots & \ddots & & \vdots \\ O & \cdots & I_{f_k} & -A_{e-1} \end{pmatrix}$$

where I_{f_k} is the $f_k \times f_k$ identity matrix and A_i is the $f_k \times f_k$ diagonal matrix with diagonal entries $a_i, \varphi(a_i), \dots, \varphi^{f_k-1}(a_i)$. Now it is an easy exercise to check that the characteristic polynomial of this matrix is $\prod_{0\le i\le f_k-1} E^{\varphi^i}(T)$ as desired.

Now we note that roots of the characteristic polynomial of $D^{K'}_{\mathrm{cris}}(k(\chi_\pi))$ are the $f_{K'/k}$-th power of those of $D^{k}_{\mathrm{cris}}(k(\chi_\pi))$ since the latter describes the action of φ^{f_k} but the former describes that of $\varphi^{f_{K'}} = \varphi^{f_{K'/k}f_k}$. Furthermore, we also note that all the roots of the right hand side of (2.5) is a conjugate of π over $\mathbb{Q}_p$. Hence, it follows from the claim (2.5) that any root of the characteristic polynomial of $D^{K'}_{\mathrm{cris}}(k(\chi_\pi))$ is of the form $\tau(\pi)^{-f_{K'/k}}$ for some $\tau \in \Gamma_k$. On the other hand, for crystalline characters $\psi_1, \psi_2 \colon G_{K'} \to k^\times$, we have a surjection $D^{K'}_{\mathrm{cris}}(k(\psi_1)) \otimes_{K'_0} D^{K'}_{\mathrm{cris}}(k(\psi_2)) \to D^{K'}_{\mathrm{cris}}(k(\psi_1\psi_2))$ induced from the natural map $k(\psi_1) \otimes_{\mathbb{Q}_p} k(\psi_2) \to k(\psi_1) \otimes_k k(\psi_2) = k(\psi_1\psi_2)$. Here, K'_0 is the maximal unramified subextension of $K'/\mathbb{Q}_p$. In particular, roots of the characteristic polynomial of $D^{K'}_{\mathrm{cris}}(k(\psi_1\psi_2))$ is a product of those of $D^{K'}_{\mathrm{cris}}(k(\psi_1))$ and $D^{K'}_{\mathrm{cris}}(k(\psi_2))$. By this fact, we know that any root of the characteristic polynomial of $D^{K'}_{\mathrm{cris}}(k(\chi_\pi^{r_\sigma}))$ is of the form $\prod_{\tau \in \Gamma_k} \tau(\pi)^{-f_{K'/k} n_\tau^\sigma}$ with $\sum_{\tau \in \Gamma_k} n^\sigma_\tau = r_\sigma$. By (2.4), the same holds for the roots of the characteristic polynomial of $D^{K'}_{\mathrm{cris}}(E(\sigma^{-1} \circ \chi_\pi^{r_\sigma}))$. Therefore, since α is a root of the characteristic polynomial of $D^{K'}_{\mathrm{cris}}(W) = D^{K'}_{\mathrm{cris}}(E(\prod_{\sigma \in \Gamma_k} \sigma^{-1} \circ \chi_\pi^{r_\sigma}))$, we have

$$\alpha = \prod_{\tau \in \Gamma_k} \tau(\pi)^{-f_{K'/k} n_\tau}$$

with $\sum_{\tau \in \Gamma_k} n_\tau = \sum_{\sigma \in \Gamma_k} r_\sigma =: R$. (Here, $n_\tau = \sum_{\sigma \in \Gamma_k} n^\sigma_\tau$.) We note that R is an integer such that $dh_1 \le R \le dh_2$ since we have $r_\sigma \in [h_1, h_2]$. This completes the proof. □

We need the following two standard lemmas which describe inclusion properties of two Lubin-Tate extensions.

Lemma 2.7. *Let k_2/k_1 be a finite extension of p-adic fields with residual extension degree f. For $i = 1, 2$, let π_i be a uniformizer of k_i and $k_{i,\pi_i}/k_i$ the Lubin-Tate extension associated with π_i.*
(1) We have $\mathrm{Nr}_{k_2/k_1}(\pi_2) = \pi_1^f$ if and only if $k_{1,\pi_1} \subset k_{2,\pi_2}$.
(2) $\pi_1^{-f}\mathrm{Nr}_{k_2/k_1}(\pi_2)$ is a root of unity if and only if there exists a finite extension $M/k_{2,\pi_2}$ such that $k_{1,\pi_1} \subset M$. If this is the case, we can take M to be the degree $\sharp\mu_\infty(k_1)$ subextension in $k_2^{\mathrm{ab}}/k_{2,\pi_2}$. Here, $\mu_\infty(k_1)$ is the set of roots of unity in k_1.

Proof. For $i = 1, 2$, we denote by k_i^{ur} and k_i^{ab} the maximal unramified extension of k_i and the maximal abelian extension of k_i, respectively. We recall that the Artin map $\mathrm{Art}_{k_i} \colon k_i^\times \to \mathrm{Gal}(k_i^{\mathrm{ab}}/k_i)$ associated with k_i satisfies $\mathrm{Art}_{k_i}(\pi_i)|_{k_{i,\pi_i}} = \mathrm{id}$ and $\mathrm{Art}_{k_i}(\pi_i)|_{k_i^{\mathrm{ur}}} = \mathrm{Frob}_{k_i}$, where Frob_{k_i} is the geometric Frobenius of k_i.
(1) Suppose $\mathrm{Nr}_{k_2/k_1}(\pi_2) = \pi_1^f$. For any lift $\sigma \in G_{k_2}$ of $\mathrm{Art}_{k_2}(\pi_2)$, we have

$$\sigma|_{k_{1,\pi_1}} = (\mathrm{Art}_{k_2}(\pi_2)|_{k_1^{\mathrm{ab}}})|_{k_{1,\pi_1}} = \mathrm{Art}_{k_1}(\mathrm{Nr}_{k_2/k_1}(\pi_2))|_{k_{1,\pi_1}} = \mathrm{Art}_{k_1}(\pi_1)^f|_{k_{1,\pi_1}} = \mathrm{id}.$$

Since the intersection of the fixed fields (in $\overline{\mathbb{Q}}_p$) of such σ's is k_{2,π_2}, we obtain the desired result.

Conversely, suppose $k_{1,\pi_1} \subset k_{2,\pi_2}$. Then we have

$$\mathrm{Art}_{k_1}(\mathrm{Nr}_{k_2/k_1}(\pi_2))|_{k_{1,\pi_1}} = \mathrm{Art}_{k_2}(\pi_2)|_{k_{1,\pi_1}} = (\mathrm{Art}_{k_2}(\pi_2)|_{k_{2,\pi_2}})|_{k_{1,\pi_1}} = \mathrm{id}$$

and

$$\mathrm{Art}_{k_1}(\mathrm{Nr}_{k_2/k_1}(\pi_2))|_{k_1^{\mathrm{ur}}} = \mathrm{Art}_{k_2}(\pi_2)|_{k_1^{\mathrm{ur}}} = (\mathrm{Art}_{k_2}(\pi_2)|_{k_2^{\mathrm{ur}}})|_{k_1^{\mathrm{ur}}} = \mathrm{Frob}_{k_2}|_{k_1^{\mathrm{ur}}} = \mathrm{Frob}_{k_1}^f.$$

Thus we have $\mathrm{Art}_{k_1}(\mathrm{Nr}_{k_2/k_1}(\pi_2)) = \mathrm{Art}_{k_1}(\pi_1^f)$, which shows $\mathrm{Nr}_{k_2/k_1}(\pi_2) = \pi_1^f$.
(2) A very similar proof to that of (1) proceeds. Suppose that $\pi_1^{-f}\mathrm{Nr}_{k_2/k_1}(\pi_2)$ is a root of unity. If we denote by h the order of the set of roots of unity in k_1, then we have $\mathrm{Nr}_{k_2/k_1}(\pi_2^h) = \pi_1^{fh}$. We see that any lift $\sigma \in G_{k_2}$ of $\mathrm{Art}_{k_2}(\pi_2^h)$ fixes k_{1,π_1}. This implies that k_{1,π_1} is contained in a degree h subextension in $k_2^{\mathrm{ab}}/k_{2,\pi_2}$.

Suppose that there exists a finite extension $M/k_{2,\pi_2}$ such that $k_{1,\pi_1} \subset M$. Then $M' := k_{1,\pi_1}k_{2,\pi_2}$ is a finite subextension in $k_2^{\mathrm{ab}}/k_{2,\pi_2}$. Put $h = [M' : k_{2,\pi_2}]$. Since $\mathrm{Art}_{k_2}(\pi_2^h)|_{M'}$ is the identity map, we have $\mathrm{Art}_{k_1}(\mathrm{Nr}_{k_2/k_1}(\pi_2^h))|_{k_{1,\pi_1}} = \mathrm{id}$ and $\mathrm{Art}_{k_1}(\mathrm{Nr}_{k_2/k_1}(\pi_2^h))|_{k_1^{\mathrm{ur}}} = \mathrm{Frob}_{k_1}^{fh}$. Thus we have $\mathrm{Art}_{k_1}(\mathrm{Nr}_{k_2/k_1}(\pi_2^h)) = \mathrm{Art}_{k_1}(\pi_1^{fh})$, which shows $\mathrm{Nr}_{k_2/k_1}(\pi_2^h) = \pi_1^{fh}$.

□

We recall that k_G is the Galois closure of $k/\mathbb{Q}_p$ and $d_G := [k_G : \mathbb{Q}_p]$.

Lemma 2.8. *There exist a finite unramified extension k'/k_G and a uniformizer π' of k' which satisfy the following.*

- $\mathrm{Nr}_{k'/k}(\pi') = \pi^{f_{k'/k}}$,
- $k_\pi \subset k'_{\pi'}$, *where $k'_{\pi'}$ is the Lubin-Tate extension of k' associated with π',*
- *the extension $k'/\mathbb{Q}_p$ is Galois, and*
- $[k' : \mathbb{Q}_p] = sd_G$ *for some integer $1 \leq s \leq e_G$.*

Proof. Let $k_{G,0}/k$ be the maximal unramified subextension in k_G/k. By [Se1, Chapter V, §6, Proposition 10], there exists an unramified extension $\tilde{k}_0$ over $k_{G,0}$ of degree at most $[k_G : k_{G,0}](= e_G)$ such that $\pi = \mathrm{Nr}_{k'/\tilde{k}_0}(\pi')$ for some $\pi' \in (k')^\times$, where $k' := k_G\tilde{k}_0$. Since $k_G/\mathbb{Q}_p$ is Galois and k'/k_G is unramified, we see that $k'/\mathbb{Q}_p$ is Galois. We also see that π' is a uniformizer of k'. Since $k_G \cap \tilde{k}_0 = k_{G,0}$, we have $[k' : k_G] = [\tilde{k}_0 : k_{G,0}] \leq e_G$. Thus we obtain $[k' : \mathbb{Q}_p] = [k' : k_G][k_G : \mathbb{Q}_p] = sd_G$ for some integer $1 \leq s \leq e_G$. Furthermore, we have $\mathrm{Nr}_{k'/k}(\pi') = \mathrm{Nr}_{\tilde{k}_0/k}(\mathrm{Nr}_{k'/\tilde{k}_0}(\pi')) = \mathrm{Nr}_{\tilde{k}_0/k}(\pi) = \pi^{f_{k'/k}}$. By Lemma 2.7, we have $k_\pi \subset k'_{\pi'}$. □

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. First we consider the case where $k/\mathbb{Q}_p$ is Galois. Replacing L by a finite extension, we may assume that L/K is Galois. Then V^{G_L} is a G_K-stable submodule of V. By Lemma 2.6, there exists a finite extension K'/Kk such that any root α of the characteristic polynomial of $D_{\mathrm{st}}^{K'}(V^{G_L})$ is of the form

$$\alpha = a^{f_{K'/k}}, \quad a = \prod_{\tau \in \Gamma_k} \tau(\pi)^{-n_\tau}$$

with some integers $(n_\tau)_{\tau \in \Gamma_k}$ such that $dh_1 \leq \sum_{\tau \in \Gamma_k} n_\tau \leq dh_2$. Here, $d := [k : \mathbb{Q}_p]$. Put $R := \sum_{\tau \in \Gamma_k} n_\tau$. Then we have

$$\prod_{\sigma \in \Gamma_k} \sigma(a) = \prod_{\tau \in \Gamma_k} \prod_{\sigma \in \Gamma_k} \sigma\tau(\pi)^{-n_\tau} = \prod_{\tau \in \Gamma_k} \mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{-n_\tau} = \mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{-R}. \tag{2.6}$$

Since V^{G_L} has Weil weights in S, we see that $\sigma(a)$ is a q-Weil number of weight $w \in S$ for any $\sigma \in \Gamma_k$. Thus it follows from the condition $w \neq 0$ and the equation (2.6) that we have $R \neq 0$. Therefore, we obtain that $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$ is a q-Weil number of weight $-w/h$ where $h := R/d \in [h_1, h_2] \cap (1/d)\mathbb{Z}$. This shows Theorem 2.3 (1). Now Theorem 2.3 (2) follows from the fact that we have $(q^r\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{-h})^d = \mathrm{Nr}_{k/\mathbb{Q}_p}(q^r a)$ and $(q^r a)^{f_{K'/k}} = q_{K'}^r \alpha$ is a root of the characteristic polynomial of $D_{\mathrm{st}}^{K'}(V(-r))$ (here, $q_{K'}$ is the order of the residue field of K'). Thus we obtained a proof of Theorem 2.3 in the case where $k/\mathbb{Q}_p$ is Galois.

Next we consider the case where $k/\mathbb{Q}_p$ is not necessarily Galois. Take a finite extension k'/k_G and a uniformizer π' of k' as in Lemma 2.8. Put $d' = [k' : \mathbb{Q}_p]$. We have $d' = sd_G$ for some $1 \leq s \leq e_G$. Let q' be the order of the residue field of k'. Let L' be the composite field of L and $k'_{\pi'}$, which is a finite extension of $Kk'_{\pi'}$. Assume that V^{G_L} is not zero. Since $V^{G_{L'}}$ is also not zero and the extension $k'/\mathbb{Q}_p$ is Galois, we know that $\mathrm{Nr}_{k'/\mathbb{Q}_p}(\pi')$ is a q'-Weil number of weight $-w/h$ for some $w \in S$ and $h \in [h_1, h_2] \cap (1/d')\mathbb{Z}$. By the equation $\mathrm{Nr}_{k'/k}(\pi') = \pi^{f_{k'/k}}$, we have $\mathrm{Nr}_{k'/\mathbb{Q}_p}(\pi') = (\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi))^{f_{k'/k}}$, and hence $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$ is a q-Weil number of weight $-w/h$. Furthermore, we have $q'^r\mathrm{Nr}_{k'/\mathbb{Q}_p}(\pi')^{-h} = (q^r\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{-h})^{f_{k'/k}}$. This completes the proof of Theorem 2.3. □

2.2 Proofs of Theorems 1.1 and 1.2

We prove Theorems 1.1 and 1.2 in the Introduction. We start with a proof of Theorem 1.2.

Proof of Theorem 1.2. Let the notation be as in the theorem. Replacing K by a finite extension, we may assume that X has good reduction over K. Then we know that V is crystalline with Hodge-Tate weights in $[-i+r, r]$ (cf. [Fa1], [Fa2]). We claim that V has Weil weight $i-2r$. Let K_0 be the maximal unramified subextension of $K/\mathbb{Q}_p$. Put $q_K = p^{f_K}$, the order of the residue field of K. Let Y be the special fiber of a proper smooth model of X over the integer ring of K. By the crystalline conjecture shown by Faltings [Fa2] (cf. [Ni], [Tsu]), we have an isomorphism $D^K_{\mathrm{cris}}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \simeq K_0 \otimes_{W(\mathbb{F}_{q_K})} H^i_{\mathrm{cris}}(Y/W(\mathbb{F}_{q_K}))$ of φ-modules over K_0. It follows from Corollary 1.3 of [CLS] (cf. [KM, Theorem 1] and [Na, Remark 2.2.4 (4)]) that the characteristic polynomial of $K_0 \otimes_{W(\mathbb{F}_{q_K})} H^i_{\mathrm{cris}}(Y/W(\mathbb{F}_{q_K}))$ coincides with $\mathrm{char}_X(T) := \det(T - \mathrm{Frob}_{\mathbb{F}_{q_K}} \mid H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ for any prime $\ell \neq p$. Hence we obtain the fact that the characteristic polynomial $\mathrm{char}_{V(-r)}(T)$ of $D^K_{\mathrm{cris}}(V(-r))$ divides $\mathrm{char}_X(T)$. Thus it follows from the Weil Conjecture (cf. [De1], [De2]) that $\mathrm{char}_{V(-r)}(T)$ has algebraic integer coefficients and its roots are q_K-Weil numbers of weight i. In particular, V has Weil weight $i-2r$ as desired. Now the result follows by Theorem 2.3. □

Finally, we prove Theorem 1.1. Let A be an abelian variety over a p-adic field K and let ℓ be any prime number. We denote by $T_\ell(A)$ the ℓ-adic Tate module of A and set $V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. It is well known that we have G_K-equivariant isomorphisms $V_\ell(A) \simeq H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)^\vee$ and $V_\ell(A)/T_\ell(A) \simeq A(\overline{K})[\ell^\infty]$. Here, $A(\overline{K})[\ell^\infty]$ is the ℓ-power torsion subgroup of $A(\overline{K})$. Furthermore, for an algebraic extension L/K, the ℓ-power torsion subgroup $A(L)[\ell^\infty]$ of $A(L)$ is finite if and only if $V_\ell(A)^{G_L} = 0$. Below we denote by L any finite extension of Kk_π. Assume that A has potential good reduction and $N_{k/\mathbb{Q}_p}(\pi)$ satisfies the condition in the statement of Theorem 1.1. For the proof of Theorem 1.1, it is enough to show that both the p-part and the prime-to-p part of $A(L)_{\mathrm{tor}}$ are finite.

Finiteness of the p-part of $A(L)_{\mathrm{tor}}$: If we put $W = V_p(A)^{G_L}$, then it is enough to show $W = 0$. Replacing L by a finite extension, we may suppose that the extension L/K is Galois. Then the G_K-action on $V_p(A)$ preserves W, and thus the dual representation $W^\vee$ of W is a quotient representation of $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$. By Theorem 1.2, we have $W^\vee = (W^\vee)^{G_L} = 0$, which implies $W = 0$ as desired.

Finiteness of the prime-to-p part of $A(L)_{\mathrm{tor}}$: The finiteness of the prime-to-p part of $A(L)_{\mathrm{tor}}$ follows from the following general property.

Proposition 2.9. *Let A be an abelian variety over K with potential good reduction. Let M be an algebraic extension of K with finite residue field. Then the prime-to-p part of $A(M)_{\mathrm{tor}}$ is finite.*

Proof. Replacing K and M by finite extensions, we may assume that A has good reduction over K. It follows from the criterion of Néron-Ogg-Shafarevich [ST, Theorem 1] that the prime-to-p part of $A(M)_{\mathrm{tor}}$ has values in the maximal unramified subextension of M/K, which is a finite extension of K by assumption on M. Then the result follows from the main theorem of [Ma]. □

Therefore, we obtained the proof of Theorem 1.1.

Remark 2.10. (This is pointed out by Yuichiro Taguchi.) We can construct an example which gives a negative answer to the question given in the Introduction for potential good reduction case. Let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the full ring of integers $\mathcal{O}_F$ of an imaginary quadratic field F. Let $\psi = \psi_{E/F}$ be the Grössencharacter associated with E. Let p be a prime number such that E has good ordinary reduction and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_F$ above p. If we set $\pi := \psi(\mathfrak{p})$, then π is a generator of $\mathfrak{p}$ and we have $p = \pi\bar{\pi}$. Here, $\bar{\pi}$ is the complex conjugate of π. Note that π is a p-Weil number of weight 1. Let $K = k$ be the completion of F at $\mathfrak{p}$. By definition, we have $K = k = \mathbb{Q}_p$ and π is a uniformizer of them. If we identify a decomposition group of G_F at $\mathfrak{p}$ with G_K, then the action of G_K on the set of π-power torsion points of $E(\overline{K})$ is given by the Lubin-Tate character χ_π associated with π. In particular, we see that $E(Kk_\pi)[p^\infty]$ is infinite.

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