Kummer-faithfulness over p-adic fields

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Abstract

The notion of a Kummer-faithful field, defined by Mochizuki, is expected as one of suitable base fields for anabelian geometry. In this paper, we study Kummer-faithfulness for algebraic extension fields of p-adic fields. We show that Kummer-faithfulness for such fields are deeply related with various finiteness properties on torsion points of (semi-)abelian varieties. For example, a Galois extension K of a p-adic field is Kummer-faithful with finite residue field if and only if, for any finite extension L of K and any abelian variety over L, its L-rational torsion subgroup is finite. In addition, we study Kummer-faithfulness for Lubin-Tate extension fields.

Contents

1 Introduction

Anabelian geometry is an area of arithmetic geometry that studies how much information concerning the geometry of certain geometric objects, so-called "anabelian varieties", can be reconstructed from various data associated with their arithmetic fundamental groups. The philosophy of anabelian geometry was first suggested by Grothendieck in *Esquisse d'un Programme* and *Brief an G. Faltings* (cf. [SL97]), and he proposed that anabelian geometry should be considered over fields that are finitely generated over their prime fields. Nowadays, Grothendieck's original conjecture for hyperbolic curves over finitely generated

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fields over the prime field has been proved by Nakamura ([Nak90b], [Nak90a]), Tamagawa ([Tam97]), Mochizuki ([Moc96], [Moc07]) and Stix ([Sti02a], [Sti02b]).

However, led by Mochizuki, it has been revealed that anabelian geometry can be developed over a broader class of fields. *Kummer-faithful fields*, defined by Mochizuki [Moc15] and the main topic of this paper, form one such class. A perfect field K is Kummer-faithful if, for every finite extension L of K and every semi-abelian variety A over L, the Mordell-Weil group $A(L)$ has a trivial divisible part; see Definition 2.2. (Precisely, in [Moc15], Kummer-faithful fields are assumed to be of characteristic zero. In [Hos17], Kummer-faithful fields are defined also for positive characteristic cases.) Kummer-faithfulness asserts the injectivity of the Kummer map associated with semi-abelian varieties; thus, roughly speaking, Kummer-faithfulness guarantees that "Kummer theory for semi-abelian varieties works effectively". Typical examples of Kummer-faithful fields are finitely generated fields over $\mathbb{Q}$, which are related to the Grothendieck's original setting. Moreover, *sub-p-adic fields* are Kummer-faithful (here, a field is sub-p-adic if it is isomorphic to a finitely generated field over $\mathbb{Q}_p$). As one of the important results related to the Grothendieck conjecture over Kummer-faithful fields, Hoshi proved in [Hos17] that a "point-theoretic" and "Galois-preserving" isomorphism between the étale fundamental groups of affine hyperbolic curves over Kummer-faithful fields arises from an isomorphism of schemes.

A study of various properties of Kummer-faifthfulness, together with the construction of examples of Kummer-faithful fields, is important for understanding the range of fields over which anabelian geometry can be developed. Moreover, in recent years, studying these topics has become a research interest in its own right. Let k be a number field and K/k an algebraic extension. In [OT22, Corollary 2.15], Taguchi and the author studied Kummer-faithfulness of K in terms of ramification theory. We showed that K is Kummer-faithful if K/k is a Galois extension with "finite maximal ramification break everywhere" (cf. Definition 2.14 of *loc. cit.*). As a typical example, the field obtained by adjoining to $\mathbb{Q}$ all ℓ-th roots of unity for all prime ℓ is Kummer-faithful. Ohtani [Oht22] and Asayama-Taguchi [AT25] studied Kummer-faithfulness for extremely large fields. Let G be the absolute Galois group of k and e a positive integer. For $\sigma = (\sigma_1, \ldots, \sigma_e) \in G^e$ denote by $\overline{k}(\sigma)$ the fixed field of σ in $\overline{k}$, and by $\overline{k}[\sigma]$ the maximal Galois subextension of k in $\overline{k}(\sigma)$. It is known that $\overline{k}[\sigma]$ is Kummer-faithful for almost all $\sigma \in G^e$ (in terms of the (normalized) Haar measure); see [Oht22, Corollary 1] for the case where $e \geq 2$, and [AT25, Theorem 5.3] for the general case. Moreover, if $e \geq 2$, $\overline{k}(\sigma)$ is Kummer-faithful for almost all $\sigma \in G^e$ (cf. [AT25, Theorem 5.2]). The structure of the Mordell-Weil group of (semi-)abelian varieties over $\overline{k}(\sigma)$ or $\overline{k}[\sigma]$ satisfies interesting properties; see Section 3 of [AT25] for more information. On the other hand, Murotani showed in [Mur23a] that an algebraic extension field $\mathbb{F}$ over $\mathbb{F}_p$ is Kummer-faithful if and only if the absolute Galois group of $\mathbb{F}$ is isomorphic to $\hat{\mathbb{Z}}$. It should be notable that he also studied Kummer-faithfulness for higher local fields (cf. [Mur23b]).

In this paper, we study Kummer-faithfulness for algebraic extension fields of p-adic fields (= finite extension fields of $\mathbb{Q}_p$). Since sub-p-adic fields are Kummer-faithful, we know that p-adic fields are Kummer-faithful (but this can be checked immediately from the main theorem of [Mat55]). If we restrict our attention to tamely ramified (or unramified) Galois extensions K of some p-adic field, we will see that Kummer-faithfulness has a simple interpretation; in fact, for such K, K is Kummer-faithful if and only if $K/\mathbb{Q}_p$ is quasi-finite (see Corollaries 3.4 and 3.7). Thus our main interest is Kummer-faithfulness for algebraic extensions of p-adic fields with infinite wild ramification. For this, we focus on Lubin-Tate extension fields of p-adic fields. For a power q of p, we say that α is a *Weil q-integer* if $|\iota(\alpha)| = \sqrt{q}$ for every embedding $\iota\colon \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$. We denote by $W(q)$ the set of Weil q-integers.

Theorem 1.1 (A part of Theorem 4.2)**.** *Let k be a p-adic field with residue field $\mathbb{F}_q$ and π a uniformizer of k. Denote by k_π the Lubin-Tate extension field of k associated with π.*

(1) *If k is a Galois extension of $\mathbb{Q}_p$ and k_π is not Kummer-faithful, then either of the following holds.*

  (a) $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$.

  (b) *For some $(r_\sigma)_{\sigma\in\Gamma_k}$ with $r_\sigma \in \{0,1\}$, it holds $\prod_{\sigma\in\Gamma_k} \sigma\pi^{r_\sigma} \in W(q)$. Here, Γ_k is the set of $\mathbb{Q}_p$-algebra embeddings $k \hookrightarrow \overline{\mathbb{Q}}_p$.*

(2) *If $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$ or $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \in W(q)$, then k_π is not Kummer-faithful.*

Applying Theorem 1.1 with $k = \mathbb{Q}_p$, we obtain

Corollary 1.2. *Assume $k = \mathbb{Q}_p$. Then the following are equivalent.*

(i) *k_π is Kummer-faithful.*

(ii) *$p^{-1}\pi \notin \mu_{p-1}$ and $\pi \notin W(p)$.*

Similar results related to assertion (1) of Theorem 1.1 have already been studied in [Oze23, Theorem 1.1], and we will essentially follow the same arguments provided there in our proof. Assertion (2), which can be seen as a partial converse of (1), is not studied in *loc. cit.* In Section 4.2, we construct an example of non-Kummer faithful k_π such that the assumptions in Theorem 1.1 (2) do not hold but (b) in Theorem 1.1 (1) holds. The author has a slight hope that k_π is not Kummer-faithful if and only if either (a) or (b) in Theorem 1.1 (1) holds.

On the other hand, in the p-adic setting, thanks to the theory of Tate curves and non-archimedian rigid uniformization theorems, we can show that Kummer-faithfulness is equivalent to certain finiteness properties of torsion points of abelian varieties.

Theorem 1.3 (Corollary of Theorem 3.9)**.** *Let K be a Galois extension of a p-adic field.*

(1) *The following are equivalent.*

  (a) *K is Kummer-faithful.*

  (b) *Any finite extension of K has only finitely many ℓ-power roots of unity for every prime ℓ, and the group $A(L)[p^\infty]$ is finite for any finite extension L/K and any abelian variety A/L with good reduction.*

(2) *The following are equivalent.*

  (a) *K is Kummer-faithful with finite residue field.*

  (b) *The torsion subgroup of $A(L)$ is finite for any finite extension L/K and any abelian variety A/L.*

The finiteness properties such as (b) of Theorem 1.3 (2) have been studied as a standard problem in arithmetic theory of abelian varieties. It seems that many known results are for abelian varieties over number fields, but some results are also known for those over p-adic fields. It is a theorem of Imai [Ima75] that, if $K = \mathbb{Q}_p(\mu_{p^\infty})$, the torsion subgroup of $A(L)$ is finite for any finite extension L/K and any abelian variety A/K with potential good reduction. This result is known to be essentially valuable in studies such as Iwasawa theory. Kubo and Taguchi [KT13] generalized Imai's theorem in the sense that Imai's theorem holds even after replacing $\mathbb{Q}_p(\mu_{p^\infty})$ with the field $k(k^{1/p^\infty})$ obtained by adjoining to a p-adic field k all p-power roots of all element of k. However, the fields K appearing here are not Kummer-faithful since Tate curves over $\mathbb{Q}_p$ contain non-trivial divisible K-rational element (see also Corollary 1.2).

It would be very interesting to consider analogues of Theorem 1.3 over global fields; however, the author currently has no idea what such statements might look like.

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Notation. A *number field* is a finite extension of the field $\mathbb{Q}$ of rational numbers. Let p be a rational prime. A *p-adic field* is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. For any field F, we fix a separable closure $\overline{F}$ of F and we denote by G_F the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F.

2 General theory for Kummer-faithful fields

In this section, we recall the definition of Kummer-faithful fields (cf. [Moc15, Def. 1.5], [Hos17, Def. 1.2]) and study some standard properties for Kummer-faithfulness in general settings. Before starting the main part, we briefly recall the degree of (not necessary finite) algebraic extensions. A *supernatural number* is a formal product $\mathfrak{n} = \prod_\ell \ell^{n_\ell}$ where ℓ runs over the set of primes and $n_\ell \in \mathbb{Z} \cup \{\infty\}$. Using the unique decomposition into prime powers, we can view any natural number as a supernatural number. The product of supernatural numbers are defined by the natural way, and also the greatest common divisors and the least common multiple of supernatural numbers. Let L be an algebraic extension of a perfect field K. We define the *extension degree* $[L : K]$ of L/K by

$$[L : K] := \mathrm{lcm}\{[K' : K] \mid K' \text{ is a finite extension of } K \text{ contained in } L\}$$

as a supernatural number. By Galois theory, we have a group theoretic interpretation of $[L : K]$ as follows. Take any Galois extension $\tilde{L}$ of K which contains L, and set $G := \mathrm{Gal}(\tilde{L}/K)$ and $H := \mathrm{Gal}(\tilde{L}/L)$. Then we see $[L : K] = (G : H)$, where the right hand side is the index of profinite groups in the sense of Section 1.3 of [Ser97]. For algebraic extensions $K \subset L \subset M$, we have $[M : K] = [M : L] \cdot [L : K]$. If an algebraic extension L/K is of the form $L = \bigcup_i K_i$ for some finite extensions K_i of K, one has $[L : K] = \mathrm{lcm}\{[K_i : K]\}_i$.

Definition 2.1. Let L be an algebraic extension of a field K with $[L : K] = \prod_\ell \ell^{n_\ell}$. We say that L/K is *quasi-finite* if n_ℓ is finite for any prime ℓ.

Every finite extensions are clearly quasi-finite. For an algebraic extension $\mathbb{E}$ of a finite field $\mathbb{F}$ with $[\mathbb{E} : \mathbb{F}] = \prod_\ell \ell^{n_\ell}$, then one sees that $\mathbb{E}/\mathbb{F}$ is quasi-finite if and only if $G_\mathbb{E}$ is isomorphic to $\hat{\mathbb{Z}}$. If L is an unramified extension of a p-adic field K, then L/K is quasi-finite if and only if the residue field extension of L/K is quasi-finite.

Now let us recall the definition of Kummer-faithful fields and study some basic properties. Let M be a $\mathbb{Z}$-module and ℓ a prime. We say that $P \in M$ is *divisible* (resp. *ℓ-divisible*) if, for any integer $n > 0$, there exists $Q \in M$ such that $P = nQ$ (resp. $P = \ell^n Q$). We denote by M_{div} (resp. $M_{\ell\text{-div}}$) the set of divisible (resp. ℓ-divisible) elements of M, that is,

$$M_{\mathrm{div}} = \bigcap_{n>0} nM, \qquad M_{\ell\text{-div}} = \bigcap_{n>0} \ell^n M.$$

Note that $P \in M$ is divisible if and only if it is ℓ-divisible for all primes ℓ. In fact, we have $M_{\mathrm{div}} = \bigcap_\ell M_{\ell\text{-div}}$. If a $\mathbb{Z}$-module M is divisible (i.e., $M_{\mathrm{div}} = M$), it is known (e.g., [Fuc15, Chapter 4, Theorem 3.1]) that M is isomorphic to $\mathbb{Q}^I \oplus \left(\oplus_p (\mathbb{Z}[1/p]/\mathbb{Z})^{I_p}\right)$, where p runs over the set of primes and I, I_p are some index sets.

Definition 2.2. A perfect field K is *Kummer-faithful* (resp. *AVKF*, resp. *torally Kummer-faithful*) if, for every finite extension L of K and every semi-abelian variety (resp. abelian variety, resp. torus) A over L, it holds that

$$A(L)_{\mathrm{div}} = 0.$$

It is clear that any subfield of a Kummer-faithful field is also Kummer-faithful, and any finite extension of a Kummer-faithful field is also Kummer-faithful. As is remarked in [Moc15, Rem. 1.5.2], by considering the Weil restriction, one verifies immediately that one obtains an equivalent definition of Kummer-faithful, if, in Definition 2.2, one restricts L to be equal to K. The same statements holds if we replace "Kummer-faithful" with "AVKF" or "torally Kummer-faithful".

It is shown by Mochizuki in Remark 1.5.4 of [Moc15] that any *sub-p-adic field* is Kummer-faithful. Here, recall that a field k is sub-p-adic if there exists a prime p and a finitely generated field extension L of $\mathbb{Q}_p$ such that k is isomorphic to a subfield of L. Let F^{cyc} be the maximal cyclotomic field of a number field F. Then it follows from the result of Ribet (and Proposition 2.6) that F^{cyc} is AVKF (however, it is not Kummer-faithful). Furthermore, if L is a quasi-finite Galois extension of F^{cyc}, then L is also AVKF (cf. [HMT25, Proposition 6.3 (i)]). Furthermore, it is shown by Murotani and the author in [MO25, Theorem 1.3] that the field $F(F^{1/\infty})$ obtained by adjoining to F all roots of all elements of F is AVKF.

Proposition 2.3. *A perfect field K is Kummer-faithful if and only if it is both torally Kummer-faithful and AVKF.*

Proof. Suppose that K is both torally Kummer-faithful and AVKF. Let L be a finite extension of L and A a semi-abelian variety over L. There exists an exact sequence $0 \to T \to A \to B \to 0$ of K-group schemes where T is a torus and B is an abelian variety, which induces an exact sequence $0 \to T(L) \to A(L) \to B(L)$ of $\mathbb{Z}$-modules. Since K is both torally Kummer-faithful and AVKF, we know that both $T(L)_{\mathrm{div}}$ and $B(L)_{\mathrm{div}}$ are zero. Hence it follows from Lemma 2.4 below that $A(L)_{\mathrm{div}}$ is also zero. □

Lemma 2.4. *Let $0 \to L \to M \to N$ be an exact sequence of $\mathbb{Z}$-modules. Assume that $L_{\mathrm{div}} = 0$, $N_{\mathrm{div}} = 0$ and the kernel of the n-th multiplication map on M is finite for any $n > 0$. Then we have $M_{\mathrm{div}} = 0$.*

Proof. Take any $x \in M_{\mathrm{div}}$. For any $n > 0$, we denote by X_n the set of all $y \in M$ such that $ny = x$. Then $\{X_n\}_{n>0}$ forms a projective system with transition maps $f_{n,m} \colon X_m \to X_n$ given by $f_{n,m}(z) = (m/n)z$ for $n \mid m$. Since each X_n is a non-empty finite set, the projective limit $\varprojlim_n X_n$ is non-empty. Take any $(x_n)_n \in \varprojlim_n X_n$ and denote by $f \colon M \to N$ the map in the statement of the lemma. Since each x_n is a divisible element of M, we see $f(x_n) \in N_{\mathrm{div}}$. This shows $f(x_n) = 0$ and thus $x_n \in L$. Since $x = nx_n \in L$ for any n, we find that x is a divisible element of L. This gives $x = 0$ as desired. □

We study some relations between Kummer-faithfulness and finiteness of prime power torsion part of semi-abelian varieties.

Definition 2.5. (1) Let ℓ be a prime. A perfect field K is *ℓ^∞-semi-AV-tor-finite* (resp. *ℓ^∞-AV-tor-finite*) if, for every finite extension L of K and every semi-abelian variety (resp. abelian variety) A over L, it holds that $A(L)[\ell^\infty]$ is finite.
(2) A perfect field K is *locally semi-AV-tor-finite* (resp. *locally AV-tor-finite*) if it is ℓ^∞-semi-AV-tor-finite (resp. ℓ^∞-AV-tor-finite) for every prime ℓ.
(3) A perfect field K is *semi-AV-tor-finite* (resp. *AV-tor-finite*) if, for every finite extension L of K and every semi-abelian variety (resp. abelian variety) A over L, it holds that $A(L)_{\mathrm{tor}}$ is finite.

It is helpful to the readers to refer [HMT25, Section 6] for various properties of Kummer-faithful fields, AVKF fields, ℓ^∞-AV-tor-finite fields and so on. Note that locally semi-AV-tor-finite is equivalent to *$\mathfrak{Primes}^\infty$-AV-tor-finite* in the sense of [HMT25, Definition 6.1 (iv)]. Any sub-p-adic field is semi-AV-tor-finite by Proposition 2.9 of [OT22]. The theorem of Ribet [KL81] shows that the maximal cyclotomic field F^{cyc} of a number field F is AV-tor-finite. Moreover, it is shown in [MO25, Corollary 1.2] that the field $F((\mathcal{O}_F^\times)^{1/\infty})$ obtained by adjoining to F all roots of all units of the integer ring of F is also AV-tor-finite.

Proposition 2.6. *Let K be an algebraic extension of a field k. Let A be a semi-abelian variety* (*resp. abelian variety, resp. torus*) *over k. Consider the following conditions.*

(a) *$A(K)_{\mathrm{div}}$ is zero.*

(b) *$A(K)[\ell^\infty]$ is finite for any prime ℓ.*

Then we have (*a*) $\Rightarrow$ (*b*)*. If k is Kummer-faithful* (*resp. AVKF, resp. torally Kummer-faithful*) *and K is a Galois extension of k, then we have* (*a*) $\Leftrightarrow$ (*b*)*.*

Proof. The statement for Kummer-faithful fields follows from Proposition 2.4 of [OT22]. The arguments in *loc. cit.* proceed also for AVKF fields and torally Kummer-faithful fields. □

Here is an immediate consequence of the proposition above.

Corollary 2.7. (1) *A perfect field is locally semi-AV-tor-finite if it is Kummer-faithful.*
(2) *For a Galois extension field of a Kummer-faithful field, it is locally semi-AV-tor-finite if and only if it is Kummer-faithful.*

Note that there exists a locally semi-AV-tor-finite field which is not Kummer-faithful. Let $a > 1$ be a natural number and take a system $(a_n)_{n>0}$ in $\overline{\mathbb{Q}}$ such that $a_1 = 1$ and $a_{nm}^m = n$ for all $n, m > 0$. Denote by K the extension field of $\mathbb{Q}$ obtained by adjoining all a_n for $n > 0$. Then, K is locally semi-AV-tor-finite but is not Kummer-faithful since $K^\times$ contains a non-trivial divisible element a.

Proposition 2.8. *Let $\Box \in \{\text{semi-AV}, \text{AV}\}$ and ℓ a prime. Let K be a perfect field and L a potentially prime-to-ℓ extension of K. If K is ℓ^∞-$\Box$-tor-finite, then L is also ℓ^∞-$\Box$-tor-finite.*

In particular, any quasi-finite extension of a locally semi-AV-tor-finite field is also locally semi-AV-tor-finite.

Proof. Let L' be a finite extension of L and A a (semi-)abelian variety over L'. Take any finite extension K'/K contained in L' such that A is defined over K'. Setting $L'' := L'(K'(A[\ell]))$, we know that $L''/K'(A[\ell])$ is potentially prime-to-ℓ since so is L/K. Thus there exists a finite extension K'' of $K'(A[\ell])$ contained in L'' such that L''/K'' is prime-to-ℓ. Since $K''(A[\ell^\infty])$ is a pro-ℓ extension of K'' but L'' is a prime-to-ℓ extension of K'', we see that the intersection $L'' \cap K''(A[\ell^\infty])$ is equal to K''. Hence we obtain $A(L')[\ell^\infty] \subset A(L'')[\ell^\infty] = A(K'')[\ell^\infty]$. Since K is ℓ^∞-semi-AV-tor-finite, the finiteness of $A(K'')[\ell^\infty]$ is assured, which gives the fact that $A(L')[\ell^\infty]$ is finite. $\Box$

It may be helpful to write down the following implications:

$$\begin{tikzcd}
\text{semi-AV-tor-finite} \arrow[r, Rightarrow] & \text{locally semi-AV-tor-finite} \arrow[r, Rightarrow] & \ell^\infty\text{-semi-AV-tor-finite} \\
\text{sub-}p\text{-adic} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{KF} = \text{TKF and AVKF} \arrow[u, Rightarrow, "(*)"'] &
\end{tikzcd}$$

Here, KF and TKF stand for Kummer-faithful and torally Kummer-faithful, respectively. The vertical arrow $(*)$ is an equivalence relation for the class of Galois extension fields of Kummer-faithful fields. It follows from Theorem 3.9 (2) below that, for Galois extension fields of p-adic fields, we have an equivalence "semi-AV-tor-finite $\Leftrightarrow$ KF with finite residue fields".

3 Kummer-faithfulness in p-adic settings

In this section, we study Kummer-faithfulness for algebraic extensions K of $\mathbb{Q}_p$. Throughout this section, we denote by $\mathbb{F}_K$ the residue field of K. First we should point out that, by the existence of Tate curves, one can check that Kummer-faithfulness is equivalent to AVKF property in this situation.

Proposition 3.1. *Let K be an algebraic extension of $\mathbb{Q}_p$.*

(1) *For a prime ℓ, K is ℓ^∞-semi-AV-tor-finite if and only if it is ℓ^∞-AV-tor-finite.*

(2) *K is Kummer-faithful if and only if K is AVKF.*

Proof. Let $E_{/\mathbb{Q}_p}$ be the Tate curve associated with uniformizing element p. To show (1), it suffices to prove that $\mu_{\ell^\infty}(L)$ is finite for any finite extension L of K under the assumption that K is ℓ^∞-AV-tor-finite, but this follows immediately from the existence of an injection $\mu_{\ell^\infty}(L) \hookrightarrow E(L)[\ell^\infty]$. To show (2), it suffices to prove that the divisible part $(L^\times)_{\mathrm{div}}$ of $L^\times$ is trivial for any finite extension L of K under the assumption that K is AVKF, but this also follows immediately from Lemma 2.4 and an exact sequence $0 \to p^{\mathbb{Z}} \to L^\times \to E(L)$ of abelian groups. $\square$

3.1 Unramified extensions and tamely ramified extensions

In this section, we give criteria of Kummer-faithfulness for unramified, or tamely ramified, extension fields of some p-adic fields. In addition, we show that Kummer-faithful fields that are Galois extensions of p-adic fields admit a decomposition of quasi-finite extension fields and (possibly of infinite degree) p-power extensions.

Recall that $\mathbb{F}_K$ is the residue field of an algebraic extension field K of $\mathbb{Q}_p$. We say that a field K is *stably μ_{ℓ^∞}-finite* if $\mu_{\ell^\infty}(L)$ is finite for any finite extension L of K.

Lemma 3.2. *Let K be an algebraic extension of $\mathbb{Q}_p$. Denote by $G^{\ell}_{\mathbb{F}_K}$ the maximal pro-ℓ quotient of $G_{\mathbb{F}_K}$ for any prime ℓ.*

(1) *Assume $\ell \neq p$. Then K is stably μ_{ℓ^∞}-finite if and only if $G^{\ell}_{\mathbb{F}_K} \simeq \mathbb{Z}_\ell$.*

(2) *If K is p^∞-semi-AV-tor-finite, then $G^{p}_{\mathbb{F}_K} \simeq \mathbb{Z}_p$.*

Proof. (1) We may suppose that K contains μ_ℓ. Then the maximal pro-ℓ extension of $\mathbb{F}_K$ is $\mathbb{F}_K(\mu_{\ell^\infty})$, and $G^{\ell}_{\mathbb{F}_K} \simeq \mathrm{Gal}(\mathbb{F}_p(\mu_{\ell^\infty})/\mathbb{F}_K \cap \mathbb{F}_p(\mu_{\ell^\infty})) \subset \mathrm{Gal}(\mathbb{F}_p(\mu_{\ell^\infty})/\mathbb{F}_p(\mu_\ell)) \simeq \mathbb{Z}_\ell$. Thus we have the following equivalent relations: $G^{\ell}_{\mathbb{F}_K} \simeq \mathbb{Z}_\ell \Leftrightarrow \#\mu_{\ell^\infty}(\mathbb{F}_K) < \infty \Leftrightarrow \#\mu_{\ell^\infty}(K) < \infty$. The result immediately follows.

(2) Assume $G^{p}_{\mathbb{F}_K} \not\simeq \mathbb{Z}_p$, that is, $G^{p}_{\mathbb{F}_K}$ is trivial. Take a CM elliptic curve E defined over a p-adic subfield k of K with the properties that E has good ordinary reduction over k and every endomorphism of E is defined over k. Let $V_p(E)$ and $V_p(\bar{E})$ be the p-adic Tate module of E and that of the reduction $\bar{E}$ of E, respectively. With a suitable choice of a basis of $V_p(E)$, the representation $\rho\colon G_k \to GL_{\mathbb{Q}_p}(V_p(E)) \simeq GL_2(\mathbb{Q}_p)$ is of the form

$$\rho = \begin{pmatrix} \chi_p\varepsilon^{-1} & u \\ 0 & \varepsilon \end{pmatrix},$$

where χ_p is the p-adic cyclotomic character, ε is an unramified character defined by the G_k-action on $V_p(\bar{E})$ and u is a continuous map. Since $G^{p}_{\mathbb{F}_K}$ is trivial, we find that an open subgroup of $G_{\mathbb{F}_K}$ acts on $\bar{E}[p^\infty]$ trivial. Replacing K by a finite extension, we may assume that $\bar{E}[p^\infty]$ is defined over $\mathbb{F}_K$. This implies $G_K \subset \ker\varepsilon$. Since ρ has an abelian image, we have $\rho(\sigma)\rho(\tau) = \rho(\tau)\rho(\sigma)$ for any $\sigma, \tau \in G_k$, which gives $(\varepsilon^{-1}(\tau)\chi_p(\tau) - \varepsilon(\tau))u(\sigma) = (\varepsilon^{-1}(\sigma)\chi_p(\sigma) - \varepsilon(\sigma))u(\tau)$. In particular, for $\sigma, \tau \in G_K$, we have

$$(\chi_p(\tau) - 1)u(\sigma) = (\chi_p(\sigma) - 1)u(\tau). \tag{3.1}$$

Take an element $\tau_0 \in G_K \smallsetminus \ker\chi_p$ (such an element exists since p^∞-semi-AV-tor-finiteness of K in particular implies K is stably μ_{p^∞}-finite). By (3.1), we have $u(\sigma) = \frac{u(\tau_0)(\chi_p(\sigma)-1)}{\chi_p(\tau_0)-1}$ for any $\sigma \in G_K$. This shows that $u = c(\chi_p - 1)$ on G_K for some $c \in \mathbb{Q}_p$. We see that the vector $\mathbf{x} = {}^t(-c, 1)$ satisfies $\rho(\sigma)\mathbf{x} = \mathbf{x}$ for any $\sigma \in G_K$. Hence $V_p(E)^{G_K}$ is not zero, that is, $E(K)[p^\infty]$ is infinite. This contradicts the assumption that K is p^∞-semi-AV-tor-finite. $\square$

Proposition 3.3. *Let K be an algebraic extension of $\mathbb{Q}_p$.*

(1) *If K is Kummer-faithful, then $\mathbb{F}_K$ is quasi-finite, that is, $G_{\mathbb{F}_K} \simeq \hat{\mathbb{Z}}$.*

(2) *If K is AV-tor-finite, then $\mathbb{F}_K$ is finite.*

Proof. (1) Since Kummer-faithful fields over p-adic fields are stably μ_{ℓ^∞}-finite for any prime ℓ and p^∞-semi-AV-tor-finite, the result follows immediately from Lemma 3.2.

(2) Assume that $\mathbb{F}_K$ is infinite. There exists an increasing extensions $\mathbb{F}_{p^{m_1}} \subset \mathbb{F}_{p^{m_2}} \subset \cdots$ of subfields of $\mathbb{F}_K$ with $m_1 < m_2 < \cdots$. This in particular implies that $\mathbb{F}_K$ contains all the $(p^{m_i}-1)$-th roots of unity for all i, and the same holds for K. Hence, for the Tate curve $E_{/\mathbb{Q}_p}$ associated with any choice of uniformizing element, $E(K)$ contains infinitely many torsion points but this contradicts the assumption that K is AV-tor-finite. □

Corollary 3.4. *Let K be an unramified extension of some p-adic field. Then, the following are equivalnet.*

(a) *K is Kummer-faithful.*

(b) *$K/\mathbb{Q}_p$ is quasi-finite.*

(c) *$\mathbb{F}_K$ is Kummer-faithful.*

Proof. It is shown by Murotani [Mur23a, Theorem B] that $\mathbb{F}_K$ is Kummer-faithful if and only if $G_{\mathbb{F}_K} \simeq \hat{\mathbb{Z}}$. By assumption on K, this is equivalent to say that $K/\mathbb{Q}_p$ is quasi-finite. Thus the result immediately follows from Propositions and 2.6, 2.8 and 3.3. □

Remark 3.5. Let L be the completion of an algebraic extension, with finite ramification, of some p-adic field. Then, it follows from [Mur23a, Proposition 3.7] that, if the residue field of L is Kummer-faithful, then L is Kummer-faithful (see also [Tsu23, Proposition 1.4]). Thus the equivalent conditions (a), (b) and (c) in Corollary 3.4 are also equivalent to the following condition:

(d) The completion of K is Kummer-faithful.

Next we consider Kummer-faithfulness for tamely ramified extensions.

Lemma 3.6. *Let K be an algebraic extension of a p-adic field k. Let M (resp. N) be the maximal unramified (resp. maximal tamely ramified) extension of k contained in K.*

(1) *If K is Kummer-faithful, then M/k is quasi-finite.*

(2) *If K is torally Kummer-faithful and N is a Galois extension of M, then N/M is quasi-finite.*

(3) *If K is Kummer-faithful and is a Galois extension of k, then N/k is quasi-finite.*

Proof. The assertion (1) follows from Corollary 3.4, and (3) follows from (1) and (2). Thus it suffices to show (2). Assume that there exists a prime ℓ such that $v_\ell([N:M]) = \infty$. Note that ℓ is not equal to p since N/M is totally tamely ramified. There exists an infinite set of finite Galois subextensions $\{M_i\}_{i\in\mathbb{M}}$ in N/M with the property that $v_\ell(e_i) < v_\ell(e_{i+1})$ where $e_i := [M_i : M]$. Since M_i is a Galois extension of M, it follows from [Lan94, Chapter II, §5, Proposition 12] that M_i contains e_i-th roots of unity. Since $\lim_{i\to\infty} v_\ell(e_i) = \infty$, it follows that N contains all ℓ-power roots of unity. This contradicts the assumption that K is torally Kummer-faithful. □

Corollary 3.7. *Let K be a Galois extension of some p-adic fields. Then, the following are equivalent.*

(a) *$K/\mathbb{Q}_p$ is tame and K is Kummer-faithful.*

(b) *$K/\mathbb{Q}_p$ is quasi-finite.*

(c) *$K/\mathbb{Q}_p$ is tame, K is torally Kummer-faithful and $\mathbb{F}_K$ is Kummer-faithful.*

Proof. $(a) \Rightarrow (b)$ follows from Lemma 3.6 (3). $(b) \Rightarrow (a)$ follows from Propositions 2.6 and 2.8. $(a), (b) \Rightarrow (c)$ follows from Corollary 3.4. $(c) \Rightarrow (b)$ follows from Corollary 3.4 and Lemma 3.6 (2). □

Proposition 3.8. *Let K be a Galois extension of a p-adic field. Then the following are equivalent.*

(a) *K is Kummer-faithful.*

(b) *K has a decomposition $K = MN$. Here,*

- *M is Kummer-faithful with (possibly infinite) p-power degree over a p-adic field, and*
- *N is a quasi-finite Galois extension of a p-adic field.*

If K is Kummer-faithful and is abelian extension of a p-adic field, then we can choose N above so that it is an unramified quasi-finite extension of a p-adic field.

Proof. $(b) \Rightarrow (a)$ follows from Proposition 2.8. We show $(a) \Rightarrow (b)$. Assume that K is Kummer-faithful. Let k be a p-adic subfield of K so that K is a Galois extension of k. Let N be the maximal tamely ramified extension of k contained in K, which is a Galois extension of k since so is K. By Lemma 3.6 (3), replacing k by a finite subextension in K, we may suppose that N/k is prime-to-p. Then, it follows from Lemma 5 of [Iwa55] that the exact sequence $1 \to \mathrm{Gal}(K/N) \to \mathrm{Gal}(K/k) \to \mathrm{Gal}(N/k) \to 1$ splits, that is, there exists an algebraic extension M of k contained in K such that $K = MN$ and $M \cap N = k$. Since K/N is a pro-p extension, we see that the supernatural number $[M : k]$ is a power of p. This finishes a proof of $(a) \Rightarrow (b)$. Finally we note that, under the assumption that K is abelian over a p-adic field k, the tame ramification index of K/k is finite. □

In view of the above proposition, the essential difficulty in constructing a Kummer-faithful field over a p-adic field lies in the problem of whether an (almost) pro-p algebraic extension which is Kummer-faithful can be constructed. Later we will study some criterion of Kummer-faithfulness for Lubin-Tate extensions of p-adic fields.

3.2 Finiteness of torsion points

The purpose of this section is to give some equivalent conditions of Kummer-faithfulness in terms of finiteness of torsion points.

Theorem 3.9. *Let $\mathbb{Q}_p \subset k \subset K$ be algebraic extensions.*

(1) *Assume that k is Kummer-faithful and K is a Galois extension of k. Then, the following are equivalent.*

(a) *K is Kummer-faithful.*

(b) *K is stably μ_{ℓ^∞}-finite for every prime ℓ, and the group $A(L)[p^\infty]$ is finite for any finite extension L/K and any abelian variety A/L with good reduction.*

(2) *Assume that k is both AV-tor-finite and Kummer-faithful, and also assume that K is a Galois extension of k. Then the following are equivalent.*

(a) *K is Kummer-faithful with finite residue field.*

(b) *K is semi-AV-tor-finite.*

(c) *K is AV-tor-finite.*

We note that the implications $(a) \Rightarrow (b)$ in Theorem 3.9 (1) and $(b) \Rightarrow (c)$ in Theorem 3.9 (2) clearly hold for *every* algebraic extension field K of $\mathbb{Q}_p$. If we remove the assumption "Galois", then $(a) \Rightarrow (b)$, (c) in Theorem 3.9 (2) does not hold (although the author does not know about the converse implication), see Remark 3.11 (2).

First we show Theorem 3.9 (1); this is an immediate consequence of Proposition 2.7, Proposition 3.2 and Lemma 3.10 below.

Lemma 3.10. *Let $\mathbb{Q}_p \subset k \subset K$ be algebraic extensions. Let ℓ be a prime. Assume that k is ℓ^∞-semi-AV-tor-finite and K is a Galois extension of k. Consider the following conditions.*

(a) *K is ℓ^∞-semi-AV-tor-finite.*

(b) *K is stably μ_{ℓ^∞}-finite, and the group $A(L)[\ell^\infty]$ is finite for any finite extension L/K and any abelian variety $A_{/L}$ with good reduction.*

(c) *K is stably μ_{ℓ^∞}-finite.*

Then, the following hold.

(1) *We have $(a) \Leftrightarrow (b) \Rightarrow (c)$.*

(2) *If ℓ is not equal to p, then we have $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.*

Proof. The equivalence $(a) \Leftrightarrow (b)$ follows from essentially the same proof as that of Corollary 2.4 (1) of [Oze23]. (We should give two remarks about this. At first, in *loc cit.*, k is assumed to be p-adic fields. However, the same argument proceeds by using ℓ^∞-semi-AV-tor-finiteness instead of Matuck's theorem. Secondly, we remark that the key for the proof is a non-archimedian rigid uniformization theorem. We will use a similar method in the proof of Theorem 3.9 (2) below.) It suffices to show $(c) \Rightarrow (b)$ if $\ell \neq p$. Let $\ell \neq p$ and assume that K is stably μ_{ℓ^∞}-finite. Let L be a finite extension of K and A an abelian variety over L with good reduction. Let us denote by $\bar{A}$ the reduction of A. Since ℓ is prime to p, the reduction map induces a bijection between $A(L)[\ell^\infty]$ and $\bar{A}(\mathbb{F}_L)[\ell^\infty]$. On the other hand, it follows from (c) that $\mathbb{F}_L$ is a potential prime-to-ℓ extension over a finite field. Thus $\bar{A}(\mathbb{F}_L)[\ell^\infty]$ is defined over a finite field. In particular, the group $\bar{A}(\mathbb{F}_L)[\ell^\infty]$, and thus also $A(L)[\ell^\infty]$, must be finite. □

Next we give a proof of Theorem 3.9 (2).

Proof of Theorem 3.9 (2). The implication $(b) \Rightarrow (c)$ is clear, and $(c) \Rightarrow (a)$ is a result of Propositions 2.6 and 3.3. We show $(a) \Rightarrow (b)$. Assume the condition (a). Let L/K be a finite extension and X/L a semi-abelian variety. The goal is to show that the torsion subgroup of $X(L)$ is finite. We will prove this in four steps depending on the situation of X.

Step 1. Consider the case where $X = T$ is a torus. Replacing L by a finite extension, we may assume that the torus T splits over L. Since L is Kummer-faithful, $T(L)[\ell^\infty]$ is finite for every prime ℓ. Moreover, one can check that $T(L)[\ell]$ is trivial for almost all ℓ: If not, we have $\mu_\ell \subset L$ for infinitely many primes ℓ but this contradicts the fact that L has a finite residue field. Hence $T(L)_{\mathrm{tor}}$ is finite.

Step 2. Consider the case where $X = A$ is an abelian variety with potential good reduction. Replacing L by a finite extension, we may suppose that A has good reduction over L. Since L is AVKF, $A(L)[\ell^\infty]$ is finite for every prime ℓ. Furthermore, since the reduction map from the prime-to-p part $A(L)_{p'}$ of $A(L)_{\mathrm{tor}}$ is injective and the residue field of L is finite, we see that $A(L)_{p'}$ is finite. Hence $A(L)_{\mathrm{tor}}$ is finite.

Step 3. Consider the case where $X = A$ is an abelian variety (without reduction hypothesis). Let g be the dimension of A. Take a p-adic subfield K_0 in L so that A is defined over K_0. Applying a non-archimedean rigid uniformization theorem to $A_{/K_0}$ ([Ray71], [BL91] and [BX96]), we find that there exist the following data, which is called a rigid uniformization of A (cf. [BX96, Definition 1.1 and Theorem 1.2]):

(i) S is a semi-abelian variety of dimension g fits into an exact sequence of K_0-group schemes $0 \to T \to S \to B \to 0$ where T is a torus of rank m and B is an abelian variety which has potential good reduction,

(ii) a closed immersion of rigid K_0-groups $N^{\mathrm{an}} \hookrightarrow S^{\mathrm{an}}$ where N is a group scheme which is isomorphic to $\mathbb{Z}^{\oplus m}$ after a finite base extension. Here, the subscript "an" is the GAGA functor, and

(iii) a faithfully flat morphism $S^{\mathrm{an}} \to A^{\mathrm{an}}$ of rigid K_0-groups with kernel N^{an}.

We have exact sequences

$$0 \to N(\overline{K}) \to S(\overline{K}) \to A(\overline{K}) \to 0 \quad \text{and} \quad 0 \to T(\overline{K}) \to S(\overline{K}) \to B(\overline{K}) \to 0$$

of G_{K_0}-modules. For the proof, replacing L and K_0 by finite extensions, we may assume that L is a Galois extension of k and N is constant over the base field K_0. We set $k_0 := kK_0$. Note that L is a Galois extension of k_0. The exact sequence $0 \to T(\overline{K}) \to S(\overline{K}) \to B(\overline{K}) \to 0$ of G_{k_0}-modules induces an exact sequence $0 \to T(L)_{\mathrm{tor}} \to S(L)_{\mathrm{tor}} \to B(L)_{\mathrm{tor}}$ of G_{k_0}-modules. By Step 1 and Step 2, it follows that $T(L)_{\mathrm{tor}}$ and $B(L)_{\mathrm{tor}}$ are finite, which shows that $S(L)_{\mathrm{tor}}$ is also finite. Since k is AV-tor-finite, there exists an integer $M > 0$ such that both $S(L)_{\mathrm{tor}} \subset S(\overline{K})[M]$ and $A(k_0)_{\mathrm{tor}} \subset A(\overline{K})[M]$ hold. Let $n > 0$ be any integer divided by M^2. Since L is a Galois extension of k_0, there exists an exact sequence $0 \to S(L)[n] \to A(L)[n] \to N(\overline{K})/nN(\overline{K})$ of G_{k_0}-modules. Let P be an element of $A(L)[n]$. Since G_{k_0} acts on $N(\overline{K})$ trivial, we know that $\sigma P - P \in S(L)[n] \subset S(\overline{K})[M]$ for any $\sigma \in G_{k_0}$. Thus we find $MP \in A(k_0)_{\mathrm{tor}} \subset A(\overline{K})[M]$, which implies $A(L)[n] \subset A(\overline{K})[M^2]$. Since this relation holds for any n divided by M^2, we obtain that $A(L)_{\mathrm{tor}}$ is contained in $A(\overline{K})[M^2]$, which must be finite.

Step 4. Finally, we consider the case where X is a semi-abelian variety. We have an exact sequence

$$0 \to T \to X \to A \to 0 \tag{3.2}$$

of group schemes over K, where T is a torus and A is an abelian variety. This induces an exact sequence $0 \to T(L)_{\mathrm{tor}} \to X(L)_{\mathrm{tor}} \to A(L)_{\mathrm{tor}}$ of abelian groups. Since $T(L)_{\mathrm{tor}}$ and $A(L)_{\mathrm{tor}}$ are finite by Step 1 and Step 3, we conclude that $X(L)_{\mathrm{tor}}$ is finite as desired. $\square$

Remark 3.11. (1) Let $\mathbb{Q}_p \subset k \subset K$ be as in Theorem 3.9 (2). The equivalent conditions (a), (b) and (c) in the theorem are also equivalent to the following condition (d):

(d) For every finite extension L of K and every commutative algebraic group G over L, it holds that $G(L)_{\mathrm{tor}}$ is finite.

In fact, we can check the equivalence between (d) and the others by almost the same method as the proof of the theorem; we should use Chevalley's decomposition of (possibly non-connected) commutative algebraic groups (cf. [Bri17, Theorem 2.9]) instead of (3.2) in the argument of Step 4.

(2) We can not remove the assumption "Galois" from the statement of Theorem 3.9 (2). Let $\mathfrak{n} = \prod_{\ell:\mathrm{prime}} \ell$ be a supernatural number and take a compatible system $(p_k)_{k|\mathfrak{n}}$ of (p^k-1)-th roots p_k of p. Let K be the extension field $\mathbb{Q}_p(p_k; k \mid \mathfrak{n})$ of $\mathbb{Q}_p$ obtained by adjoining all p_k for all $k \mid \mathfrak{n}$. We see that K is totally ramified over $\mathbb{Q}_p$ and the Galois closure $\hat{K}$ of K over $\mathbb{Q}_p$ is quasi-finite over $\mathbb{Q}_p$. The field $\hat{K}$ is locally semi-AV-tor-finite by Proposition 2.8. In particular, $\hat{K}$ is Kummer faithful by Proposition 2.7 (2) and thus so is K. On the other hand, K is not AV-tor-finite since there are infinitely many K-rational torsion points of the Tate curve with uniformizing parameter p.

4 Kummer-faithfulness of Lubin-Tate extensions

The aim of this section is to give some criteria on Kummer-faithfulness of Lubin-Tate extensions of p-adic fields. Throughout this section, we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Unless otherwise specified, all algebraic extension fields of $\mathbb{Q}_p$ appearing in this section are subfields of our fixed $\overline{\mathbb{Q}}_p$. Let k be a p-adic field and π a uniformizer. Let F_π be the Lubin-Tate formal group over the integer ring of k associated with π. Let k_π be the extension of k obtained by adjoining all π-power torsion points of F_π, which is called the Lubin-Tate extension of k associated with π. Local class field theory asserts that k_π is a maximal totally ramified abelian extension of k, and the composite of k_π and the maximal unramified extension field k^{ur} of k coincides with the maximal abelian extension field k^{ab} of k. We should remark that the following conditions on the Lubin-Tate extension field k_π are equivalent by Theorem 3.9 (2):

(a)) k_π is Kummer-faithful.

(b) k_π is semi-AV-tor-finite.

(c) k_π is AV-tor-finite.

We are interested in determining which Lubin–Tate extension fields k_π satisfy the three equivalent conditions mentioned above. Before the main part of this section, we note that it is easy to check whether k_π is torally Kummer-faithfulness or not.

Proposition 4.1. *k_π is torally Kummer-faithful if and only if $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \notin \mu_{p-1}$.*

Proof. One verifies immediately that k_π is torally Kummer-faithful if and only if it is stably μ_{p^∞}-finite by Proposition 2.6. The latter condition is equivalent to $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \notin \mu_{p-1}$ by local class field theory. □

4.1 Criterion for Lubin-Tate extensions

In this section, we give a proof of Theorem 1.1 in the Introduction. In fact, we prove more precise criterion for Kummer-faithfulness of Lubin-Tate extension fields. To simplify statements, we use the following notation. Let k be a p-adic field and π a uniformizer of k. For a tuple $(r_\sigma)_{\sigma\in\Gamma_k}$ with $r_\sigma \in \{0,1\}$, we often set

$$\hat{\pi} := \prod_{\sigma\in\Gamma_k} \sigma\pi^{r_\sigma}$$

(of course this notation depends on the choices of π and $(r_\sigma)_{\sigma\in\Gamma_k}$). We recall that $W(q)$ is the set of Weil q-integers.

Theorem 4.2. *Let k be a p-adic field with residue cardinality $q = p^f$ and π a uniformizer of k. We denote by k_π the Lubin-Tate extension field of k associated with π.*

(1) *If k is a Galois extension of $\mathbb{Q}_p$ and k_π is not Kummer-faithful, then either of the following holds.*

  (a) $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$.

  (b) *For some $(r_\sigma)_{\sigma\in\Gamma_k}$ with $r_\sigma \in \{0,1\}$, it holds $\hat{\pi} \in W(q)$.*

(2) *If $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$ or $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \in W(q)$, then k_π is not Kummer-faithful.*

(3) *Assume that k is a Galois extension of $\mathbb{Q}_p$, and also assume that all the following conditions hold.*

  (i) *For some $(r_\sigma)_{\sigma\in\Gamma_k}$ with $r_\sigma \in \{0,1\}$, it holds $\hat{\pi} \in W(q)$.*

  (ii) *$\pi^r \in \mathbb{Q}_p$ for some integer $r \geq 1$.*

  (iii) *Let $\hat{\pi}$ be as in* (i).

    1. *f divides $\mathrm{ord}_v(\hat{\pi})f_v$ for any finite place v of $\mathbb{Q}(\hat{\pi})$ above p. Here, ord_v is the valuation associated with v normalized by $\mathrm{ord}_v(\mathbb{Q}(\hat{\pi})^\times) = \mathbb{Z}$ and f_v is the residue degree of $\mathbb{Q}(\hat{\pi})/\mathbb{Q}$ at v.*
    2. *Let $f(X) \in \mathbb{Q}[X]$ be the minimal polynomial of $\hat{\pi}$ over $\mathbb{Q}$. Then, the degree of any irreducible factor of $f(X)$ in $\mathbb{Q}_p[X]$ is not of the form $n\cdot[\mathbb{Q}_p(\hat{\pi}) : \mathbb{Q}_p]$ for any integer $n \geq 2$.*

  Then, k_π is not Kummer-faithful.

The aim of this section is to prove the above theorem. Our argument relies heavily on both Honda–Tate theory and p-adic Hodge theory. We begin by briefly reviewing the aspects of these theories that are essential for our proof. Let A be a simple abelian variety over $\mathbb{F}_q$ of dimension $g > 0$ (here, "simple" stands for not "$\overline{\mathbb{F}}_q$-simple" but "$\mathbb{F}_q$-simple"). Put $D = \mathrm{End}_{\mathbb{F}_q}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and denote by $\pi_A \in D$ the geometric Frobemius endomorphism of $A_{/\mathbb{F}_q}$. Weil showed that π_A is a Weil q-integer. The degree $2g$ characteristic polynomial $f_{\bar{A}/\mathbb{F}_q} \in \mathbb{Z}[T]$ of π_A has a single monic irreducible factor over $\mathbb{Q}$, and $f_{\bar{A}/\mathbb{F}_q}$ coincides with the characteristic polynomial the action of the arithmetic q-Frobenius element of $G_{\mathbb{F}_q}$ on the ℓ-adic Tate module $V_\ell(A)$ of A for any prime $\ell \neq p$.

Theorem 4.3. *Let $\mathbb{F}$ be the finite field of order q.*

(1) *The assignment $A \mapsto \pi_A$ defines a bijection from the set of isogeny classes of simple abelian varieties over $\mathbb{F}$ to the set of $G_{\mathbb{Q}}$-conjugacy classes of Weil q-integers.*

(2) *Let A be an abelian vareity over $\mathbb{F}$ which is $\mathbb{F}$-isogenous to a power of an $\mathbb{F}$-simple abelian variety. Then there exist a finite extension $\mathbb{F}'/\mathbb{F}$ and a p-adic field k' with residue field $\mathbb{F}'$ such that A is $\mathbb{F}'$-isogenous to the reduction of a CM abelian variety with good reduction over k'.*

The first assertion is a central result of Honda–Tate theory, while the second is commonly referred to as the "Honda–Tate CM lifting theorem". For more details, we refer the reader to Section 1.6 of [CCO14].

Next, we summarize some fundamental facts about p-adic Hodge theory. For an introduction to the basic notions of p-adic Hodge theory, it is helpful for the reader to refer [Fon94a] and [Fon94b]. Let K be a p-adic field. Below we often use the following notations. Let B_{cris} be the Fontaine's p-adic period ring and set $D^K_{\mathrm{cris}}(V) := (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ for any $\mathbb{Q}_p$-representation V of G_K. Let us denote by K_0 the maximal unramified subextension of $K/\mathbb{Q}_p$. Since $B^{G_K}_{\mathrm{cris}} = K_0$, $D^K_{\mathrm{cris}}(V)$ is a K_0-vector space. Denote by φ_{K_0} the arithmetic Frobenius map of K_0, that is, the (unique) lift of the p-th power map on the residue field of K_0. This extends to B_{cris}, and moreover this extended Frobenius map induces a K_0-semi-liniear endomorphism φ on $D^K_{\mathrm{cris}}(V)$, which is also called Frobenius endomorphism. Note that the f_K-th iterate φ^{f_K} of φ on $D^K_{\mathrm{cris}}(V)$ is K_0-linear where f_K is the residue degree of $K/\mathbb{Q}_p$. It is known that $D^K_{\mathrm{cris}}(V)$ is of finite dimensional over K_0 with $\dim_{K_0} D^K_{\mathrm{cris}}(V) \leq \dim_{\mathbb{Q}_p} V$. We say that V is crystalline if the equality $\dim_{K_0} D^K_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$ holds. Let us recall locally algebraic theory for crystalline characters; see [Ser98, Appendix of Chapter III] and [Con11, Appendix B] for more precise information. Let F be a p-adic field and $\psi\colon G_K \to F^\times$ a continuous character. We denote by $F(\psi)$ the $\mathbb{Q}_p$-representation of G_K underlying a 1-dimensional F-vector space endowed with an F-linear action by G_K via ψ. We say that ψ is crystalline if $F(\psi)$ is crystalline. Suppose that ψ is crystalline. For any $\sigma \in \Gamma_F$, let $\chi_{\sigma F}\colon I_{\sigma F} \to \sigma F^\times$ be the restriction to the inertia $I_{\sigma F}$ of the Lubin-Tate character associated with any choice of uniformizer of σF (it depends on the choice of a uniformizer of σF, but its restriction to the inertia subgroup does not). Assume that K contains the Galois closure of $F/\mathbb{Q}_p$. Then, we have

$$\psi = \prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \chi^{h_\sigma}_{\sigma F} \tag{4.1}$$

on the inertia I_K for some integer h_σ where $\Gamma_F := \mathrm{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)$. Note that $\{h_\sigma \mid \sigma \in \Gamma_F\}$ is the (multi-)set of Hodge-Tate weights of $F(\psi)$, that is, $C \otimes_{\mathbb{Q}_p} F(\psi) \simeq \oplus_{\sigma \in \Gamma_F} C(h_\sigma)$ where C is the completion of $\overline{\mathbb{Q}}_p$.

Here we recall a remarkable observation of Conrad for crystalline characters. We denote by $\underline{K}^\times$ the Weil restriction $\mathrm{Res}_{K/\mathbb{Q}_p}(\mathbb{G}_m)$. This is an algebraic torus such that, for a $\mathbb{Q}_p$-algebra R, the R-valued points $\underline{K}^\times(R)$ of $\underline{K}^\times$ is $\mathbb{G}_m(R \otimes_{\mathbb{Q}_p} K)$. If $\psi\colon G_K \to F^\times$

is a continuous character, we may regard ψ as a character of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ where K^{ab} is the maximal abelian extension of K. We denote by $\psi_K\colon K^\times \to F^\times$ the composite of the reciprocity map $K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ of local class field theory and $\psi\colon \mathrm{Gal}(K^{\mathrm{ab}}/K) \to F^\times$. For example, if $\chi\colon G_K \to \mathbb{Q}_p^\times$ is the p-adic cyclotomic character, we have $\chi_K(x) = \left(\mathrm{Nr}_{K/\mathbb{Q}_p}(x)|\mathrm{Nr}_{K/\mathbb{Q}_p}|_p\right)^{-1}$ for $x \in K^\times$ where $|\cdot|_p$ is the p-adic absolute value normalized by $|p|_p = p^{-1}$. The following result is shown in Proposition B.4 of [Con11].

Proposition 4.4. *Let $\psi\colon G_K \to F^\times$ be a continuous character.*

(1) *$F(\psi)$ is crystalline if and only if there exists a (necessarily unique) $\mathbb{Q}_p$-homomorphism $\psi_{\mathrm{alg},K}\colon \underline{K}^\times \to \underline{F}^\times$ such that ψ_K and $\psi_{\mathrm{alg},K}$ (on $\mathbb{Q}_p$-points) coincides on $\mathcal{O}_K^\times (\subset \underline{K}^\times(\mathbb{Q}_p))$.*

(2) *Assume that $F(\psi)$ is crystalline and let $\psi_{\mathrm{alg},K}$ be as in (1). Then, $D^K_{\mathrm{cris}}(F(\psi^{-1}))$ is free of rank 1 over $K_0 \otimes_{\mathbb{Q}_p} F$ and its K_0-linear endomorphism φ^{f_K} is given by the action of the product*
$$\psi_K(\pi_K)\cdot\psi^{-1}_{\mathrm{alg},K}(\pi_K) \in F^\times,$$
where π_K is any uniformizer of K and f_K is the residue degree of $K/\mathbb{Q}_p$. Note that this product is independent of the choice of π_K.

Motivated by the above proposition, for a crystalline character $\psi\colon G_K \to F^\times$ of G_K, we set
$$\Phi_K(\psi) := \psi_K(\pi_K)\cdot\psi^{-1}_{\mathrm{alg},K}(\pi_K) \in F^\times.$$
This is invariant under the coefficient field extension F'/F. For a finite extension K'/K, we set $\Phi_{K'}(\psi) := \Phi_{K'}(\psi|_{G_{K'}}) \in F^\times$. For example, one verifies $\Phi_K(\chi_\pi) = \pi^{f_{K/k}}$ where $\chi_\pi\colon G_k \to k^\times$ is the Lubin-Tate character of k associated with a uniformizer π and K is a finite extension of k with residue degree $f_{K/k}$ (see also Lemma 4.5 (2) below). Note that the character $\psi_{\mathrm{alg},K}$ on $\mathbb{Q}_p$-points coincides with $\prod_{\sigma\in\Gamma_F}\sigma^{-1}\circ\mathrm{Nr}^{-h_\sigma}_{K/\sigma F}$ if K contains the Galois closure of $F/\mathbb{Q}_p$ and ψ is of the form (4.1). By Proposition 4.4 (2), one can verify that the roots of the characteristic polynomial of the φ^{f_K}-action on the K_0-vector space $D^K_{\mathrm{cris}}(F(\psi^{-1}))$ are the $\mathbb{Q}_p$-conjugates of $\Phi_K(\psi)$, that is,
$$\det(T - \varphi^{f_K} \mid D^K_{\mathrm{cris}}(F(\psi^{-1}))) = \prod_{\sigma\in\Gamma_F}(T - \sigma\Phi_K(\psi)).$$

We summarize some basic properties of $\Phi_K(\psi)$.

Lemma 4.5. *Let $\psi\colon G_K \to F^\times$ be a crystalline character and $\{h_\sigma\}_{\sigma\in\Gamma_F}$ the (multi-)set of Hodge-Tate weights of $F(\psi)$.*

(0) *The φ^{f_K}-action on the free $K_0\otimes_{\mathbb{Q}_p}F$-module $D^K_{\mathrm{cris}}(F(\psi^{-1}))$ of rank 1 is given $\Phi_K(\psi)$.*

(1) *For a crystalline character $\psi'\colon G_K \to F^\times$, we have $\Phi_K(\psi\psi') = \Phi_K(\psi)\Phi_K(\psi')$.*

(2) *For a finite extension K'/K with residue degree $f_{K'/K}$, we have $\Phi_{K'}(\psi) = \Phi_K(\psi)^{f_{K'/K}}$.*

(3) *For $\sigma \in \Gamma_F$, we have $\Phi_K(\sigma\psi) = \sigma\Phi_K(\psi)$.*

(4) *$\Phi_K(\psi) \in \mathcal{O}_F^\times$ if and only if $\sum_{\sigma\in\Gamma_F} h_\sigma = 0$.*

(5) *Assume either "$h_\sigma \geq 0$ for all σ" or "$h_\sigma \leq 0$ for all σ". If $\Phi_K(\psi) = 1$, then $\psi = 1$.*

(6) *Let π_F be a uniformizer of F with the property that $\pi_F^r \in \mathbb{Q}_p$ for some $r > 0$ (such π_F exists if $F/\mathbb{Q}_p$ is tamely ramified). If $K \supset F$, F is a Galois extension of $\mathbb{Q}_p$ and $\Phi_K(\psi) = 1$, then we have*

$$\psi = \prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \chi_{\pi_F}^{h_\sigma}$$

on an open subgroup of G_K. Moreover,

(i) *$\psi\colon G_K \to F^\times$ extends to G_F, and*

(ii) *there exists a character $\epsilon\colon G_F \to \overline{\mathbb{Q}}_p^\times$ such that $\epsilon^r = 1$ and $\psi = \epsilon \cdot \prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \chi_{\pi_F}^{h_\sigma}$ on G_F.*

Proof. (0) is Proposition 4.4 (2). (1) and (3) are clear from the definition of Φ_K. We consider (2). By local class field theory, $\psi_{K'}\colon (K')^\times \to F^\times$ is the composite of the norm map $\mathrm{Nr}_{K'/K}\colon K'^\times \to K^\times$ and $\psi_K\colon K^\times \to F^\times$, and also $\psi_{\mathrm{alg},K'}\colon \underline{K'}^\times \to \underline{F'}^\times$ is the composite of the norm map $\mathrm{Nr}_{K'/K}\colon \underline{K'}^\times \to \underline{K}^\times$ and $\psi_{\mathrm{alg},K}\colon \underline{K}^\times \to \underline{F}^\times$. Take uniformizers π_K and $\pi_{K'}$ of K and K', respectively. Writing $f = f_{K'/K}$ and $\mathrm{Nr}_{K'/K}(\pi_{K'}) = \pi_K^f u$ for some $u \in \mathcal{O}_K^\times$, we have $\Phi_{K'}(\psi) = \psi_{K'}(\pi_{K'}) \cdot \psi_{\mathrm{alg},K'}^{-1}(\pi_{K'}) = \psi_K(\pi_K^f u) \cdot \psi_{\mathrm{alg},K}^{-1}(\pi_K^f u) = (\psi_K(\pi_K) \cdot \psi_{\mathrm{alg},K}^{-1}(\pi_K))^f = \Phi_K(\psi)^f$. This shows (2). Next we show (4). By (2), we may assume that K contains the Galois closure of $F/\mathbb{Q}_p$. Thus ψ is of the form (4.1) on I_K. Note that $\Phi_K(\psi) \in \mathcal{O}_F^\times$ if and only if $\psi_{\mathrm{alg},K}(\pi_K) \in \mathcal{O}_F^\times$ since ψ_K has values in $\mathcal{O}_F^\times$. Since $\psi_{\mathrm{alg},K}(\pi_K)$ coincides with $\prod_{\sigma \in \Gamma_M} \sigma^{-1}\mathrm{Nr}_{K/\sigma F}(\pi_K)^{-h_\sigma}$, we see that $\psi_{\mathrm{alg},K}(\pi_K)$ is a p-adic unit if and only if $\sum_{\sigma \in \Gamma_F} h_\sigma = 0$, which shows (4). We show (5). It follows from the assumption and (4) that $h_\sigma = 0$ for all σ. Thus $\psi_{\mathrm{alg},K}$ is trivial and then ψ_K is trivial on $\mathcal{O}_K^\times$. Furthermore, $\Phi_K(\psi) = 1$ implies $\psi_K(\pi_K) = 1$. Hence we have $\psi_K = 1$ on $K^\times$, equivalently, $\psi = 1$. Finally, we show (6). By $\Phi_K(\psi) = 1$, we have

$$\psi_K(x) = \prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \mathrm{Nr}_{K/F}(x)^{-h_\sigma} \tag{4.2}$$

if x is any uniformizer π_K of K. It follows from the definition of $\{h_\sigma\}_{\sigma \in \Gamma_F}$ that (4.2) also holds for any $x \in \mathcal{O}_K^\times$. Hence the equality (4.2) holds for every $x \in K^\times$. Thus $\psi\colon G_K \to F^\times$ extends to G_F so that

$$\psi_F(x) = \prod_{\sigma \in \Gamma_F} \sigma^{-1} x^{-h_\sigma}$$

for $x \in F^\times$. Taking π_F as in the statement of (6), by (4), we see

$$\psi_F(\pi_F^r) = \prod_{\sigma \in \Gamma_F} (\sigma^{-1}\pi_F^r)^{-h_\sigma} = \prod_{\sigma \in \Gamma_F} \pi_F^{-rh_\sigma} = 1.$$

Hence we have $\psi^r = \left(\prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \chi_{\pi_F}^{h_\sigma}\right)^r$ on G_F. Now the result immediately follows. $\square$

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. (1) The assertion is an immediate consequence of [Oze23, Theorem 1.2]. However, we include a proof here for the sake of completeness (in fact, arguments will be simpler since we only need to consider abelian varieties here). Assume that k is a Galois extension of $\mathbb{Q}_p$ and k_π is not Kummer-faithful. By Theorem 3.9, we know that (i) k_π is not stably μ_{p^∞}-finite, or, (ii) the group $A(L)[p^\infty]$ is infinite for some finite extension L/k_π and some abelian variety $A_{/L}$ with good reduction. In the former case (i), we have $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$. In the rest of the proof we assume the latter case (ii) and let L/k_π and $A_{/L}$ be as in (ii). Take a finite subextension K/k in L/k so that $L = Kk_\pi$, A is defined over K and A has good reduction over K. Since L is a Galois extension of K and the group $A(L)[p^\infty]$ is infinite, $V := V_p(A)^{G_L}$ is a non-zero G_K-stable submodule of the p-adic Tate module $V_p(A)$ of A. Since the G_K-action on V factors through an abelian quotient, by taking a p-adic field F large enough, we have an isomorphism

$$(V \otimes_{\mathbb{Q}_p} F)^{\mathrm{ss}} \simeq F(\psi_1) \oplus F(\psi_2) \oplus \cdots \oplus F(\psi_t)$$

of $F[G_K]$-modules for some continuous characters $\psi_i \colon G_K \to F^\times$. Here, "ss" stands for the semi-simplification as a $F[G_K]$-module. Each ψ_i is crystalline since A has good reduction (see [Fon82, Section 6]; see also [CI99, Theorem 1]). Furthermore, each ψ_i factors through $\mathrm{Gal}(L/K)$. Replacing L, K and F by finite extensions, by Lemma 2.5 of [Oze23], we find that $\psi_i = \prod_{\sigma\in\Gamma_k} \sigma \circ \chi_\pi^{r_{i,\sigma}}$ on G_K for some $r_{i,\sigma} \in \{0,1\}$ (here, we need the assumption that k is a Galois extension of $\mathbb{Q}_p$ to apply the lemma). Thus

$$\Phi_K(\psi_i) = \prod_{\sigma\in\Gamma_k} \sigma\Phi_K(\chi_\pi)^{r_{i,\sigma}} = \left(\prod_{\sigma\in\Gamma_k} \sigma\pi^{r_{i,\sigma}}\right)^{f_{K/k}}. \tag{4.3}$$

Now we set

$$f(T) := \det(T - \varphi^{f_K} \mid D^K_{\mathrm{cris}}(V^\vee))$$

where f_K is the residue degree of $K/\mathbb{Q}_p$ and "$\vee$" stands for the dual of $\mathbb{Q}_p[G_K]$-modules. We have

$$f(T)^{[F:\mathbb{Q}_p]} = \det(T - \varphi^{f_K} \mid D^K_{\mathrm{cris}}(V^\vee \otimes_{\mathbb{Q}_p} F)) = \prod_{i=1}^{t} \prod_{\sigma\in\Gamma_F} (T - \sigma\Phi_K(\psi_i))$$

by Proposition 4.4 (2). The polynomial $f(T)$ is a divisor of $f_A(T) := \det(T - \varphi^{f_K} \mid D^K_{\mathrm{cris}}(V_p(A)^\vee))$ and it follows from p-adic Hodge theory (cf. [Fal89] and [CLS98]) that

$$f_A(T) = \det(T - \varphi^{f_K} \mid D^K_{\mathrm{cris}}(H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)) = \det(T - \mathrm{Frob}_K^{-1} \mid H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)) \tag{4.4}$$

for any prime $\ell \neq p$, where Frob_K stands for the arithmetic Frobenius of K. By the Weil conjecture, we conclude that each $\Phi_K(\psi_i)$ is a Weil q_K-integer. It follows from (4.3) that $\prod_{\sigma\in\Gamma_k} \sigma\pi^{r_{i,\sigma}}$ is a Weil q-integer.

(2) If $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$ is a root of unity, then k_π is not torally-Kummer faithful by Proposition 4.1 and thus it is not Kummer-faithful. We consider the case where $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \in W(q)$. It suffices to show that there exists a p-adic field K and an abelian variety defined over K which has infinitely many Kk_π-rational p-power torsion points. Let $\bar{A}$ be the simple abelian variety defined over $\mathbb{F}_q$ (uniquely determined up to $\mathbb{F}_q$-isogeny) which corresponds to the Weil q-number $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$ via Honda-Tate theory (cf. Theorem 4.3 (1)). (Note that

$\bar{A}$ is ordinary by the first Theorem of [WM71, Section III].) Moreover, by Honda-Tate lifting theorem (cf. Theorem 4.3 (2)), there exists a finite extension K/k and a CM abelian variety B over K with good reduction such that $\bar{A}$ is isogenous to the reduction $\bar{B}$ of B over the residue field of K. Put $g = \dim B$ and denote by L the CM field of B (so there exists an embedding from L into $\mathrm{End}_K(B) \otimes_{\mathbb{Z}} \mathbb{Q}$). Let $\prod_i L_i$ denote the decomposition of $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$ into a finite product of finite extensions of $\mathbb{Q}_p$ (note that, a priori, each factor field L_i does not live in the fixed algebraically closed field $\overline{\mathbb{Q}}_p$). Since the G_K-action on $V_p(B)$ commutes with $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$-action, the above decomposition gives the decomposition $V_p(B) \simeq \oplus_i V_i$ of G_K-modules with the property that each V_i is naturally equipped with a structure of one dimensional L_i-representation of G_K. By choosing a $\mathbb{Q}_p$-algebra embedding from L_i into our fixed $\overline{\mathbb{Q}}_p$ for each i, we may regard L_i as a subfield of $\overline{\mathbb{Q}}_p$. Denote by $\psi_i \colon G_K \to L_i^\times$ the character obtained by the L_i-linear G_K-action on V_i. Each ψ_i is crystalline since B has good reduction (note that the L_i-representation $L_i(\psi_i)$ considered as a $\mathbb{Q}_p$-representation is isomorphic to V_i, which is a $\mathbb{Q}_p$-subrepresentation of $V_p(B)$). It holds that

$$\det(T - \varphi^{f_K} \mid D^K_{\mathrm{cris}}(V_i^\vee)) = \prod_{\tau \in \Gamma_{L_i}} (T - \Phi_K(\tau \circ \psi_i)) \tag{4.5}$$

by Proposition 4.4 (2) and Lemma 4.5 (3). Now we set

$$f_B(T) := \det(T - \varphi^{f_K} \mid D^K_{\mathrm{cris}}(V_p(B)^\vee)).$$

Note that $f_B(T)$ is equal to the characteristic polynomial $\det(T - \mathrm{Frob}_K \mid V_\ell(B)) = \det(T - \mathrm{Frob}_K \mid V_\ell(\bar{B}))$ for any prime $\ell \neq p$ (see (4.4)), which also coincides with the characteristic polynomial $\det(T - \mathrm{Frob}_K \mid V_\ell(\bar{A}))$. Hence $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{f_{K/k}}$ is a root of $f_B(T)$ by the choice of $\bar{A}$. Since $f_B(T)$ is the product of the polynomial (4.5) over all i, we have an equality $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{f_{K/k}} = \Phi_K(\tau \circ \psi_i)$ for some i and some $\tau \in \Gamma_{L_i}$. Since $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$ is an element of $\mathbb{Q}_p$, we have $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{f_{K/k}} = \Phi_K(\psi_i)$ by Lemma 4.5 (3). Note that $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{f_{K/k}} = \left(\Phi_k(\mathrm{Nr}_{k/\mathbb{Q}_p} \circ \chi_\pi)\right)^{f_{K/k}} = \Phi_K(\mathrm{Nr}_{k/\mathbb{Q}_p} \circ \chi_\pi)$ since $\Phi_k(\chi_\pi) = \pi$. Hence we obtain

$$\Phi_K(\psi_i^{-1} \cdot (\mathrm{Nr}_{k/\mathbb{Q}_p} \circ \chi_\pi)) = 1. \tag{4.6}$$

On the other hand, replacing K by a finite extension so that K contains the Galois closure of $L_i/\mathbb{Q}_p$, we find $\psi_i = \prod_{\sigma \in \Gamma_{L_i}} \sigma^{-1} \circ \chi_{\sigma L_i}^{r_\sigma}$ for some $r_\sigma \in \{0, 1\}$ on I_K. Furthermore, we find $\mathrm{Nr}_{k/\mathbb{Q}_p} \circ \chi_\pi = \prod_{\sigma \in \Gamma_{L_i}} \sigma^{-1} \circ \chi_{\sigma L_i}$ as characters from I_K to $L_i^\times$. Then we have an equality $\psi_i^{-1} \cdot (\mathrm{Nr}_{k/\mathbb{Q}_p} \circ \chi_\pi) = \prod_{\sigma \in \Gamma_{L_i}} \sigma^{-1} \circ \chi_{\sigma L_i}^{1-r_\sigma}$ on I_K and this in particular implies that all the Hodge-Tate weights of $L_i(\psi_i^{-1} \cdot (\mathrm{Nr}_{k/\mathbb{Q}_p} \circ \chi_\pi))$ are non-negative. By (4.6) and Lemma 4.5 (5), we obtain $\psi_i = \mathrm{Nr}_{k/\mathbb{Q}_p} \circ \chi_\pi$ on G_K. Thus the Galois group G_{Kk_π} acts on V_i trivial. This in particular implies that $V_p(B)^{G_{Kk_\pi}}$ is not zero, which is equivalent to say that $B(Kk_\pi)[p^\infty]$ is infinite. Therefore, k_π is not Kummer-faithful.

(3) First we consider the case where $\hat{\pi}^2 = q$. Calculating the p-adic valuation on both sides, we have $2\sum_{\sigma \in \Gamma_k} r_\sigma = [k : \mathbb{Q}_p]$. On the other hand, taking the norm, we also have $\mathrm{Nr}_{k/\mathbb{Q}_p}(\hat{\pi})^2 = q^{[k:\mathbb{Q}_p]}$. Since k is a Galois extension of $\mathbb{Q}_p$, we have $\mathrm{Nr}_{k/\mathbb{Q}_p}(\hat{\pi})^2 = \mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)^{2\sum_{\sigma \in \Gamma_k} r_\sigma}$. Hence we see that $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$ is a root of unity, and hence k_π is not Kummer-faithful by (2).

In the rest of the proof, we assume $\hat{\pi}^2 \neq q$. Then $\mathbb{Q}(\hat{\pi})$ can not be embedded into $\mathbb{R}$. Moreover, $\mathbb{Q}(\hat{\pi})$ must be a CM field since it is a totally imaginary quadratic extension of the totally real number field $\mathbb{Q}(\hat{\pi} + q/\hat{\pi})$. The proof below is based on the method used in the proof of (2) but we need more caferul treatments. The notable difference here is that we apply a CM lifting theorem due to Chai-Conrad-Oort [CCO14], which refines the Honda–Tate lifting theorem. Let $\bar{A}$ be the simple abelian variety defined over $\mathbb{F}_q$ which corresponds to the Weil q-number $\hat{\pi}$ via Honda-Tate theory. We put $g = \dim \bar{A}$, $D = \mathrm{End}_{\mathbb{F}_q}(\bar{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $Z = \mathbb{Q}(\hat{\pi})$. Since $\hat{\pi}$ is (a $\mathbb{Q}$-conjugation of) the q-Frobenius map of $\bar{A}$, Z is a subfield of D. It is a theorem of Tate (cf. Corollary 1.6.2.2 (3) of [CCO14]) that Z is a center of D, D is a division algebra over Z of degree d^2 for some integer $d > 0$, and $2g = d \cdot [Z : \mathbb{Q}]$. Tate moreover showed that there exists a maximal subfield L in D of degree d over Z which is a CM field. Now we claim that $L = Z$. For any field F, we denote by $\mathrm{Br}(F)$ the Brauer group of F. Let $[D] \in \mathrm{Br}(Z)$ be the class of D. We denote by $\mathrm{inv}_v \colon \mathrm{Br}(Z_v) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ the local invariant map of $\mathrm{Br}(Z_v)$ for any finite place v of Z. Here, Z_v is the completion of Z at v. It is known (cf. Corollary 1.6.2.2 (3) of [CCO14]) that $\mathrm{inv}_v([D]) = 0$ if v is not above p and $\mathrm{inv}_v([D]) = \frac{\mathrm{ord}_v(\hat{\pi}) f_v}{f} \mod \mathbb{Z}$ if v is above p. By the assumption 1 of (iii) and the fact that Z has no real infinite place, we find that $[D]$ is trivial; in fact, the local invariant maps induces an injection $\mathrm{Br}(Z) \hookrightarrow \oplus_v \mathrm{Br}(Z_v)$ where v runs though all finite places of Z. On the other hand, the order of $[D]$ in $\mathrm{Br}(Z)$ coincides with the square root d of $[D : Z]$ by Theorem 1.2.4.4 of [CCO14]. Thus we have $d = 1$ and the claim follows. Consequently, we obtain the fact that $\bar{A}$ has complex multiplication over $\mathbb{F}_q$ by the CM field $L = Z$.

Let $\mathbb{Q}_q$ be the unramified extension of $\mathbb{Q}_p$ of degree f. By Theorem 4.1.1 of [CCO14] (and the constduction of "D" in page 246–247 of *loc. cit.*), there exists a finite totally ramified extension $K/\mathbb{Q}_q$ and an abelian variety B over K with good reduction such that $\bar{A}$ is $\mathbb{F}_q$-isogenous to the reduction of B. Moreover, we can take B so that B has complex multiplication over K by the same choosed CM field as $\bar{A}$; thus, we may suppose that B has complex multiplication by L. Let $\prod_i L_i$ denote the decomposition of $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$ into a finite product of finite extensions of $\mathbb{Q}_p$ and let $V_p(B) \simeq \oplus_i V_i$ be the corresponding decomposition of G_K-modules. Each V_i is naturally equipped with a structure of one dimensional L_i-representation of G_K. By choosing a $\mathbb{Q}_p$-algebra embedding from L_i into our fixed $\overline{\mathbb{Q}}_p$ for each i, we may regard L_i as a subfield of $\overline{\mathbb{Q}}_p$. We denote by $\psi_i \colon G_K \to L_i^{\times}$ the character obtained by the G_K-action on V_i. By a similar argument of the proof of (2), we know that the set of the roots of $\det(T - \mathrm{Frob}_K \mid V_\ell(\bar{A}))$ is $\{\Phi_K(\tau \circ \psi_i) \mid \tau \in \Gamma_{L_i}, i\}$. Furthermore, $\hat{\pi}$ is also a root of this polynomial since K is totally ramified over $\mathbb{Q}_q$. Thus we obtain

$$\Phi_K(\tau \circ \psi_i) = \hat{\pi} \tag{4.7}$$

for some i and some $\mathbb{Q}_p$-algebra embedding $\tau \colon L_i \hookrightarrow \overline{\mathbb{Q}}_p$. Since the left hand side of (4.7) is contained in τL_i, we have $\tau L_i \supset \mathbb{Q}_p(\hat{\pi})$. In particular, we have $[L_i : \mathbb{Q}_p] = n \cdot [\mathbb{Q}_p(\hat{\pi}) : \mathbb{Q}_p]$ for some integer $n \geq 1$. Since $[L_i : \mathbb{Q}_p]$ coincides with the degree of some irreducible factor of $f(X)$ in $\mathbb{Q}_p[X]$, it follows from the assumption 2 of (iii) that $n = 1$, that is,

$$\tau L_i = \mathbb{Q}_p(\hat{\pi}). \tag{4.8}$$

Note that (4.7) implies $\Phi_K(\tau \circ \psi_i) = \Phi_k\left(\prod_{\sigma \in \Gamma_k} \sigma \circ \chi_{\pi}^{r_\sigma}\right)$. Taking f_{Kk}-th power of this equality, we obtain $\Phi_{Kk}((\tau \circ \psi_i)^{f_K}) = \Phi_{Kk}\left(\left(\prod_{\sigma \in \Gamma_k} \sigma \circ \chi_{\pi}^{r_\sigma}\right)^{f_k}\right)$. Here we remark that, by (4.8), we may consider $\tau \circ \psi_i$ as a crystalline character of G_K with values in $k^{\times}$.

Combining this with the assumption that k is a Galois extension of $\mathbb{Q}_p$ and the assumption (ii), it follows from Lemma 4.5 (6) that we have

$$(\tau \circ \psi_i)^{f_K} = \prod_{\sigma \in \Gamma_k} \sigma \circ \chi_\pi^{m_\sigma}$$

on $G_{K'}$ for some finite extension K' of Kk and some integers m_σ. Thus the character $\tau \circ \psi_i$ restricted to $G_{K'k_\pi}$ has values in the set of f_K-th roots of unity, and hence ψ_i is trivial on $G_{K''k_\pi}$ for some finite extension K'' of K', which in particular implies that $B(K''k_\pi)[p^\infty]$ is infinite. Therefore, we conclude that k_π is not Kummer-faithful. □

Theorem 4.2 naturally leads us to consider the following question.

Question. Let k be a Galois extension of $\mathbb{Q}_p$. Are the following conditions equivalent?

(i) k_π is not Kummer-faithful.

(ii) Either of the following holds:

  (a) $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$.

  (b) For some $(r_\sigma)_{\sigma \in \Gamma_k}$ with $r_\sigma \in \{0, 1\}$, it holds $\hat{\pi} := \prod_{\sigma \in \Gamma_k} \sigma\pi^{r_\sigma} \in W(q)$.

The implications (i) $\Rightarrow$ (ii) and (ii-a) $\Rightarrow$ (i) were shown in Theorem 4.2 (1) and Proposition 4.1, respectively. Thus the remaining problem is to determine whether (ii-b) always implies (i). At the moment, the author does not have an answer for this problem. Theorem 4.2 (2) gives a partial answers for this; if $\hat{\pi}$ is the norm $\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$, then $\hat{\pi} \in W(q)$ implies (i). It is natural to ask whether there actually exists an example in which k_π is not Kummer-faithful, even though $q^{-1}\mathrm{Nr}_{k/\mathbb{Q}_p}(\pi) \notin \mu_{p-1}$, $\hat{\pi} \neq \mathrm{Nr}_{k/\mathbb{Q}_p}(\pi)$ and $\hat{\pi} \in W(q)$. By applying Theorem 4.2 (3), we construct such examples in the next section.

4.2 Examples of non-Kummer-faithful Lubin-Tate extension fields

Here we give an example of a non-Kummer-faithful Lubin-Tate extension field by applying Theorem 4.2 (3). Let r be a divisor of $p-1$. Let $F_1, F_2, \ldots, F_r$ be imaginary quadratic fields such that p splits completely by principal ideals for each F_i and $\mathrm{Gal}(F/\mathbb{Q}) \simeq \prod_{i=1}^r \mathrm{Gal}(F_i/\mathbb{Q})$, where F is the composite field of $F_1, F_2, \ldots, F_r$. Let $\mathfrak{p}$ be a finite place of F above p. Denote by $\mathfrak{p}_i$ the finite place of F_i below $\mathfrak{p}$ and take a generator ω_i of $\mathfrak{p}_i$. Let ω_i^c be the complex conjugate of ω_i; we have $\omega_i^c = \omega_i^{-1}p$. We set

$$\pi_0 := \omega_1 \cdot \prod_{i=2}^{r} \omega_i^c \quad \text{and} \quad \pi := \pi_0^{1/r}.$$

We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ with respected to $\mathfrak{p}$. With this embedding, ω_1 is a uniformizer of $\mathbb{Q}_p$ and $\omega_2^c, \ldots, \omega_r^c$ are p-adic units. Set $k := \mathbb{Q}_p(\pi)$. Then k is a totally ramified extension of $\mathbb{Q}_p$ of degree r and π is its uniformzer. Since r divides $p-1$, k is a Galois extension of $\mathbb{Q}_p$. The minimal polynomial $f(X)$ of π over $\mathbb{Q}$ is a divisor of $\prod_{i=1}^r \prod_{\sigma_i \in \{1,c\}} (X^r - \omega_1^{\sigma_1} \cdots \omega_r^{\sigma_r})$ and each $\omega_1^{\sigma_1} \cdots \omega_r^{\sigma_r}$ is of the form $p^a \omega_1^{\pm 1} \cdots \omega_r^{\pm 1}$. Hence any irreducible factor in $\mathbb{Q}_p[X]$ of $f(X)$ is a divisor of some $X^r - \omega_1^{\sigma_1} \cdots \omega_r^{\sigma_r}$. By Theorem 4.2 (3), we obtain that k_π is not Kummer-faithful.

Proposition 4.6. *Let $\mathbb{Q}_p \subset k_0 \subset k$ be finite extensions. Let π_0 and π be uniformizers of k_0 and k, respectively. Denote by k_{0,π_0} and k_π the Lubin-Tate extensions of k_0 and k associated with π_0 and π, respectively. Let f be the residue degree of k/k_0. If $\pi_0^{-f}\mathrm{Nr}_{k/k_0}(\pi)$ is a root of unity and k_{0,π_0} is not Kummer-faithful, then k_π is not Kummer-faithful.*

Proof. The result follows from local class field theory; if $\pi_0^{-f}\mathrm{Nr}_{k/k_0}(\pi)$ is a root of unity, then a finite extension of k_π contains k_{0,π_0}. □

Let k_0 be a p-adic field and π_0 a uniformizer of k_0 so that k_{0,π_0} is not Kummer-faithful. If we choose k and π by one of the following manner, it follows from Proposition 4.6 that k_π is not Kummer-faithful.

(i) Let π be a n-th root of unity of π_0 for an integer $n > 0$ and set $k := k_0(\pi)$.

(ii) Let k be a finite unramified extension of k_0 and take a uniformizer π of k such that $\pi_0^{-f}\mathrm{Nr}_{k/k_0}(\pi) = 1$. (The existence of such π is assured by, for example, [Ser68, Chapter V, §.2, Corollary of Proposition 3].)

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