4 Applications to Nonlinear Problems

4.1 Comparisom Principle for a single reaction diffusion equation

- In this section, we will apply the maximum principles for some nonlinear problems, in particular, a certain class of reaction-diffusion equations. Let us first consider the following equation:

$$u_t = \Delta u + f(u), \quad 0 < t < T, \ x \in U,$$

where $T > 0$, $U \subset \mathbb{R}^N$ and $f \in C^1(\mathbb{R})$.

Lemma 4.1

Let $U \subset \mathbb{R}^N$ be a domain, $T > 0$ and $D = (0, T) \times U$. Suppose that $c \in L^\infty(D) \cap C(\overline{D})$ and $u \in C(\overline{D}) \cap C^{1,2}(D) \cap L^\infty(D)$ satisfies

$$\begin{aligned} &u_t \le \Delta u + c(t, x)u, \quad (t, x) \in D, \\ &u(t, x) \le 0, \quad 0 < t \le T, \ x \in \partial U, \\ &u(0, x) \le 0, \quad x \in \overline{U}. \end{aligned} \tag{4.1}$$

Then it holds that $u(t, x) \le 0$ in D.

Remark: It is not necessary to impose that $c(t, x) \ge 0$ in D. This lemma is still valid if $\partial_t - \Delta$ is replaced by a general uniformly parabolic operator.
Proof:

- Let $M > \sup_{(t,x)\in D} |c(t, x)|$ and $v(t, x) = e^{Mt}u(t, x)$, that is $u(t, x) = e^{-Mt}u(t, x)$. Then v satisfies

$$\begin{aligned} v_t - \Delta v - (c(t, x) + M)v &= e^{Mt}\{Mu + u_t - \Delta u - (c(t, x) + M)u\} \\ &= e^{Mt}\{u_t - \Delta u - c(t, x)u\} \le 0 \end{aligned}$$

and clearly $v(t, x) \le 0$ on $\partial_p D$. We remark that $c(t, x) + M \ge 0$ in D.

- If U is bounded, from Theorem 3.10, we obtain $v(t, x) \le 0$, that is, $u(t, x) \le 0$ in D.

- If U is unbounded, from Theorem 3.18, we obtain $v(t, x) \le 0$ that is, $u(t, x) \le 0$ in D. □

Remark: When U is a bounded domain, if condition "$u^+(t, x) \ge u^-(t, x)$ for $0 < t \le T$, $x \in \partial U$" is replaced by "$\dfrac{\partial u^-}{\partial \nu} \le \dfrac{\partial u^+}{\partial \nu}$ for $0 < t \le T$, $x \in \partial U$", then we also obtain $u(t, x) \le 0$ in D.

Theorem 4.2(Comparison Principle)

Let $U \subset \mathbb{R}^N$ be a domain, $T > 0$ and $D = (0,T) \times U$, $f \in C^1(\mathbb{R})$. Suppose that $u^+, u^- \in C(\mathbb{R}) \cap C^{1,2}(D) \cap L^\infty(\mathbb{R})$ satisfy

$$\begin{aligned} &u_t^+ - \Delta u^+ - f(u^+) \geq u_t^- - \Delta u^- - f(u^-), \quad (t,x) \in D, \\ &u^+(t,x) \geq u^-(t,x), \quad (t,x) \in \partial_p D. \end{aligned}$$

Then it holds that $u^+(t,x) \geq u^-(t,x)$ in D.

Proof:

- Because u^-, u^+ are bounded, there exist $M > m > 0$ such that
$$m \leq u^-(t,x), u^+(t,x) \leq M$$
holds for $(t,x) \in D$.
- Let us define $w(t,x) := u^-(t,x) - u^+(t,x)$ and
$$c(t,x) := \begin{cases} \dfrac{f(u^-(t,x)) - f(u^+(t,x))}{u^-(t,x) - u^+(t,x)} & \text{if } \ u^-(t,x) \neq u^+(t,x), \\ f'(u^-(t,x)) & \text{if } \ u^-(t,x) = u^+(t,x). \end{cases}$$
- We note that $c \in C(\overline{D}) \cap L^\infty(D)$.
- From w satisfies
$$\begin{aligned} w_t - \Delta w &= u_t^- - u_t^+ - (\Delta u^- - \Delta u^+) \\ &= (u_t^- - \Delta u^-) - (u_t^+ - \Delta u^*) \\ &\leq f(u^-(t,x)) - f(u^+(t,x)) = c(t,x)w \end{aligned}$$
Hence $w_t - \Delta w - c(t,x)w \leq 0$ for $(t,x) \in D$. Moreover $w(t,x) \leq 0$ holds for $(t,x) \in \partial_p D$.
- Therefore by Lemma 4.1, we obtain that $w(t,x) \leq 0$, that is, $u^-(t,x) \leq u^+(t,x)$ for $(t,x) \in D$.$\square$

Remark: When U is a bounded domain, if condition "$u(t,x) \leq 0$ for $0 < t \leq T$, $x \in \partial U$" is replaced by "$\dfrac{\partial u}{\partial \nu} \leq 0$ for $0 < t \leq T$, $x \in \partial U$", then we also obtain $u(t,x) \leq 0$ in D.

- Consider the initial-boundary value problem
$$\begin{aligned} &u_t = \Delta u + f(u), \quad (t,x) \in D = (0,T) \times U, \\ &u(t,x) = g(t,x), \quad x \in \partial U, \\ &u(0,x) = u_0(x), \quad x \in \overline{U}. \end{aligned} \tag{4.2}$$
where $g \in C((0,T] \times \partial U)$, $u_0 \in C(\overline{U})$ are given functions (when $U = \mathbb{R}^N$ we do not impose any boundary condition in the second line).

- We say $u^+ \in C(\overline{U}) \cap C^{1,2}(D)$ is an **upper solution** of (4.2) if

$$
\begin{aligned}
&u_t^+ \geq \Delta u^+ + f(u^+), \quad (t,x) \in D = (0,T) \times U, \\
&u^+(t,x) \geq g(t,x), \quad 0 < t \leq T, \ x \in \partial U, \\
&u^+(0,x) \geq u_0(x), \quad x \in \overline{U}.
\end{aligned}
$$

- We say $u^- \in C(\overline{D}) \cap C^{1,2}(D)$ is a **lower solution** of (4.2) if

$$
\begin{aligned}
&u_t^- \leq \Delta u^- + f(u^-), \quad (t,x) \in D = (0,T) \times U, \\
&u^-(t,x) \leq g(t,x), \quad 0 < t \leq T, \ x \in \partial U, \\
&u^-(0,x) \leq u_0(x), \quad x \in \overline{U}.
\end{aligned}
$$

4.2 Comparisom Principle for a system of reaction diffusion equations

- We next consider the following reaction-diffusion system:

$$
\begin{aligned}
u_t &= d_1 \Delta u + f(u,v), \quad t > 0, \quad x \in U, \\
v_t &= d_2 \Delta v + g(u,v), \quad t > 0, \quad x \in U,
\end{aligned}
\tag{4.3}
$$

  where d_1, $d_2 > 0$ are constants and f, g are smooth functions.

Lemma 4.3

Let $U \subset \mathbb{R}^N$ be a domain, $T > 0$ and $D = (0,T) \times U$. Suppose that $u, v \in C(\mathbb{R}) \cap C^{1,2}(D) \cap L^\infty(\mathbb{R})$ satisfy

$$
\begin{aligned}
&u_t \leq d_1 \Delta u + c_{11}(t,x) u + c_{12}(t,x) v \quad (t,x) \in D, \\
&v_t \leq d_2 \Delta v + c_{21}(t,x) u + c_{22}(t,x) v \quad (t,x) \in D, \\
&u(t,x), v(t,x) \leq 0, \quad 0 < t \leq T, \ x \in \partial U, \\
&u(0,x), v(0,x) \leq 0, \quad x \in \overline{U},
\end{aligned}
$$

with some $c_{ij} \in C(\overline{D}) \cap L^\infty(D)$ $(i,j = 1,2)$. If $c_{12}(t,x) \geq 0$, $c_{21}(t,x) \geq 0$ in D. Then it holds that $u(t,x), v(t,x) \leq 0$ in D.

Proof:

- Suppose that there exists $(t_0, x_0) \in D$ such that $u(t_0, x_0) > 0$ or $v(t_0, x_0) > 0$.
- Take $M > 0$ so that $c_{j1}(t,x) + c_{j2}(t,x) \leq M$ in D holds.
- Define $U := u - \varepsilon e^{Mt}$, $V := v - \varepsilon e^{Mt}$ for small $\varepsilon > 0$.
- Take $\varepsilon > 0$ so that $U(t_0, x_0) > 0$, $V(t_0, x_0) > 0$ holds.

- Define $t_\varepsilon := \inf\{t \geq 0 : {}^\exists(t,x) \in [0,t_0) \times U \;\text{ s.t. } U(t,x) > 0 \;\text{ or }\; V(t,x) > 0\}$.
- Since $U(0,x) < 0$ and $V(t,x) < 0$, $t_\varepsilon > 0$ and there exists $x_\varepsilon \in U$ such that $U(t_\varepsilon, x_\varepsilon) = 0$ or $V(t_\varepsilon, x_\varepsilon) = 0$ and

$$U(t,x) < 0, \quad V(t,x) < 0 \quad (t,x) \in [0,t_\varepsilon) \times U.$$

- Suppose $U(t_\varepsilon, x_\varepsilon) = 0$. On $(0, t_\varepsilon] \times U$ we have

$$\begin{aligned} U_t - d_1 \Delta U - c_{11} U &= u_t - d_1 \Delta u - c_{11} u - \varepsilon M e^{Mt} + c_{11} \varepsilon e^{Mt} \\ &= c_{12} v - c_{11} u - \varepsilon M e^{Mt} + c_{11} \varepsilon e^{Mt} \\ &= c_{12} V - \varepsilon e^{Mt}(M - c_{11} - c_{12}) < 0. \end{aligned}$$

- We also have $U < 0$ on $\partial_p((0,t_\varepsilon) \times U)$. By Lemma 4.1 and Corollary 3.12(strong maximum principle) we obtain

$$U(t,x) < 0 \text{ for } \;(t,x) \in [0, t_\varepsilon] \times D,$$

which is a contradiction to $U(t_\varepsilon, x_\varepsilon) = 0$. We can get a contradiction in the case where $V(t_\varepsilon, x_\varepsilon) = 0$. $\square$

Remark: When U is a bounded domain, if condition "$u(t,x), v(t,x) \leq 0$ for $0 < t \leq T$, $x \in \partial U$" is replaced by "$\dfrac{\partial u}{\partial \nu} \leq 0$, $\dfrac{\partial v}{\partial \nu} \leq 0$ for $0 < t \leq T$, $x \in \partial U$", then we also obtain $u(t,x), v(t,x) \leq 0$ in D.

- In general, the comparison principle does not hold for reaction-diffusion system. However under some conditions on f and g, some comparison principle is available. We assume the following conditions:

$$\frac{\partial f}{\partial v} \geq 0, \quad \frac{\partial g}{\partial u} \geq 0. \tag{4.4}$$

Theorem 4.4

Let $U \subset \mathbb{R}^N$ be a domain, $T > 0$ and $D = (0,T) \times U$. Suppose that $u^{\pm}, v^{\pm} \in C(\overline{D}) \cap C^{1,2}(D)$ satisfy

$$\begin{aligned}
&(u^+)_t - d_1\Delta(u^+) + f(u^+, v^+) \geq (u^-)_t - \Delta u^- - f(u_-, v_-), \quad (t,x) \in D,\\
&(v^+)_t - d_2\Delta(v^+) + g(u^+, v^+) \geq (v^-)_t - \Delta v^- - g(u_-, v_-), \quad (t,x) \in D,\\
&u^+(t,x) \geq u^-(t,x), \quad v^+(t,x) \geq v^-(t,x) \quad 0 < t \leq T, \ x \in \partial U,\\
&u^+(0,x) \geq u^-(0,x), \quad v^+(0,x) \geq v^-(0,x) \quad x \in \overline{U}.
\end{aligned}$$

and moreover there exists $0 < m < M$ such that

$$m \leq u^{\pm}(t,x), v^{\pm}(t,x) \leq M \quad \text{for } (t,x) \in D$$

and (4.4) holds on $[m,M] \times [m,M]$. Then we have

$$u^+(t,x) \geq u^-(t,x), \quad v^+(t,x) \geq v^-(t,x) \quad \text{for} \quad (t,x) \in D.$$

Proof:

- Let $u = u^- - u^+$, $v = v^- - v^+$. Then u and v satisfy

$$\begin{aligned}
&u_t \leq d_1\Delta u + c_{11}(t,x)u + c_{12}(t,x)v \quad (t,x) \in D,\\
&v_t \leq d_2\Delta v + c_{21}(t,x)u + c_{22}(t,x)v \quad (t,x) \in D,\\
&u(t,x), v(t,x) \leq 0, \quad 0 < t \leq T, \ x \in \partial U,\\
&u(0,x), v(0,x) \leq 0, \quad x \in \overline{U},
\end{aligned}$$

  where

$$\begin{aligned}
c_{11} &= \int_0^1 f_u(\theta u^- + (1-\theta)u^+, \theta v^- + (1-\theta)v^+)d\theta,\\
c_{12} &= \int_0^1 f_v(\theta u^- + (1-\theta)u^+, \theta v^- + (1-\theta)v^+)d\theta,\\
c_{21} &= \int_0^1 g_u(\theta u^- + (1-\theta)u^+, \theta v^- + (1-\theta)v^+)d\theta,\\
c_{22} &= \int_0^1 g_v(\theta u^- + (1-\theta)u^+, \theta v^- + (1-\theta)v^+)d\theta.
\end{aligned}$$

  Here it should be noted that $c_{12}, c_{21} \geq 0$ and c_{ij} are bounded continuous functions.

- Therefore we can use Lemma 4.3 to conclude that $u^- \leq u^+$ and $v^- \leq v^+$. $\square$

Remark: When U is a bounded domain, if condition "$u^+(t,x) \geq u^-(t,x)$ and $v^-(t,x) \leq v^+(t,x)$ for $0 < t \leq T$, $x \in \partial U$" is replaced by "$\dfrac{\partial u^-}{\partial \nu} \leq \dfrac{\partial u^+}{\partial \nu}$ and $\dfrac{\partial v^-}{\partial \nu} \leq \dfrac{\partial v^+}{\partial \nu}$ for $0 < t \leq T$, $x \in \partial U$", then we also obtain $u^-(t,x) \leq u^+(t,x)$ and $v^-(t,x) \leq v^+(t,x)$ in D.

4.3 Invariant Rectangle

- Let U be a bounded domain. Consider the following reaction-diffusion system:

$$\begin{aligned}
&u_t = d_1\Delta u + f(u,v), \quad t>0, \quad x\in U,\\
&v_t = d_2\Delta v + g(u,v), \quad t>0, \quad x\in U,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad t>0, \quad x\in \partial U.\\
&u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x\in U.
\end{aligned} \tag{4.5}$$

  and corresponding ODE system:

$$\begin{aligned}
\frac{dU}{dt} &= f(U,V),\\
\frac{dV}{dt} &= g(U,V).
\end{aligned} \tag{4.6}$$

- $K\subset \mathbb{R}^2$ is said to be a **positively invariant region** of (4.6) if

  $(U(0),V(0))\in K \;\Rightarrow\; (U(t),V(t))$: solution to (4.6) satisies $(U(t),V(t))\in K (t\geq 0)$.

- $K\subset \mathbb{R}^2$ is said to be a **positively invariant region** of (4.5) if

  $(u_0(x),v_0(x))\in K (x\in U)$

  $\Rightarrow\; (u(t,x),v(t,x))$: solution to (4.5) satisies $(u(t,x),v(t,x))\in K \;\; (x\in U,\; t\geq 0)$.

Proposition 4.5

(1) If $f(a,v)\geq 0$ for any $v\in\mathbb{R}$, then $\{(u,v): u\geq a\}$ is a invariant region of (4.5).

(2) If $f(a,v)\leq 0$ for any $v\in\mathbb{R}$, then $\{(u,v): u\leq a\}$ is a invariant region of (4.5).

(3) If $g(u,c)\geq 0$ for any $u\in\mathbb{R}$, then $\{(u,v): v\geq c\}$ is a invariant region of (4.5).

(4) If $g(u,c)\leq 0$ for any $u\in\mathbb{R}$, then $\{(u,v): v\leq c\}$ is a invariant region of (4.5).

Proof:

- We only give the proof of (1).
- Let $w(t,x) = a - u(t,x)$. Then we obtain

$$\begin{aligned}
w_t = -u_t &= -d_1\Delta u - f(u,v)\\
&\leq d_1\Delta w + f(a,v) - f(u,v)\\
&= d_1\Delta w + c(t,x)w,
\end{aligned}$$

  where

$$c(t,x) = \int_0^1 f_u(\theta a + (1-\theta)u),v)d\theta.$$

- By Lemma 4.1, we obtain $u(t,x) \geq a$ for $t \geq 0$. $\square$