3 Maximum Principles for Parabolic Equations

3.1 Maximum Principle for the Heat Equation

3.1.1 Weak maximum principle for the heat equation

- In this subsection we consider the following **heat equation**, one of the parabolic equations,

$$u_t - \Delta u = 0 \tag{3.1}$$

in $D = (0,T) \times U$, where $T > 0$ and $U \subset \mathbb{R}^N$ is a domain. Here

$$(0,T) \times U := \{(t,x) \in \mathbb{R}^{N+1} : 0 < t < T, x \in U\}.$$

- For $D = (0,T) \times U$ we define its **parabolic boundary** $\partial_p D$ as follows:

$$\partial_p D = \{(t,x) : t = 0, x \in \overline{U}\} \cup \{(t,x) : 0 < t \leq T, x \in \partial U\}.$$

- Let us define function space $C^{1,2}(D)$ for the solution to parabolic equations:

$$C^{1,2}(D) := \{u = u(t,x) : u_t, u_{x_i}, u_{x_i x_j} \in C(D) \ \text{ for } \ i,j = 1, \cdots, N\}.$$

Theorem 3.1(Weak maximum principle for the heat equations)

Let U be a bounded domain and let $D = (0,T) \times U$ and let $u \in C(\overline{D}) \cap C^{1,2}(D)$ be a solution of (3.1). Then

$$\max_{(t,x)\in\overline{D}} u(t,x) = \max_{(t,x)\in\partial_p D} u(t,x)$$

and

$$\min_{(t,x)\in\overline{D}} u(t,x) = \min_{(t,x)\in\partial_p D} u(t,x)$$

hold.

Proof: We prove only for max.

- It is enough to prove that

$$\max_{(t,x)\in\overline{D}} u(t,x) \leq \max_{(t,x)\in\partial_p D} u(t,x) \tag{3.2}$$

since $\partial_p D \subset \overline{D}$.

- Since U is bounded, there exists $R > 0$ such that $U \subset B_R(0)$, where

$$B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}.$$

- For any $\varepsilon > 0$ define

$$v(t,x) = u(t,x) + \varepsilon |x|^2.$$

- By direct calculation we have

$$v_t - \Delta v = (u_t - \Delta u) + \varepsilon(\partial_t |x|^2 - \Delta |x|^2) = -2\varepsilon N < 0.$$

- Take any $\tau \in (0,T)$ and consider $K_\tau = (0,\tau] \times U$. We note that v does not achieve $\max_{(t,x)\in\overline{K_\tau}} v(t,x)$ at any point of K_τ. In fact, if v takes it maximum over $\overline{K_\tau}$ for some $(t_0,x_0) \in K_\tau$, then we have $v_t(t_0,x_0) \geq 0$ and $\Delta v(t_0,x_0) \leq 0$ and thus $v_t(t_0,x_0) - \Delta v(t_0,x_0) \geq 0$, which is a contradiction to $v_t - \Delta v < 0$.

- Therefore we obtain

$$\begin{aligned} \max_{(t,x)\in\overline{K_\tau}} u(t,x) \leq \max_{(t,x)\in\overline{K_\tau}} v(t,x) &= \max_{(t,x)\in\partial_p K_\tau} v(t,x) \\ &\leq \max_{(t,x)\in\partial_p D} v(t,x) \leq \max_{(t,x)\in\partial_p D} u(t,x) + \varepsilon R^2 \end{aligned} \tag{3.3}$$

  Here we have used the fact $\max_{(t,x)\in\partial_p K_\tau} u(t,x) \leq \max_{(t,x)\in\partial_p D} u(t,x)$.

- Since we can show that

$$\lim_{\tau\uparrow T} \max_{(t,x)\in\overline{K_\tau}} u(t,x) = \max_{(t,x)\in\overline{D}} u(t,x), \tag{3.4}$$

  by letting $\tau \uparrow T$ in (3.3) we get

$$\max_{(t,x)\in\overline{D}} u(t,x) \leq \max_{(t,x)\in\partial_p D} u(t,x) + \varepsilon R^2.$$

- By letting $\varepsilon \to 0$ we obtain (3.2).

- It remains to prove (3.4). We note that $m(\tau) := \max_{(t,x)\in\overline{K_\tau}} u(t,x)$ is nondecreasing in τ. Let $m = \max_{(t,x)\in\overline{D}} u(t,x)$.

- Suppose $m = u(t_1,x_1)$ for some $(t_1,x_1) \in \overline{D}$. If $0 \leq t_1 < T$, then $m(t) = m$ for $t \geq t_1$ and (3.4) is obvious.

- Now we assume $t_1 = T$. Take any $\eta > 0$. Since u is continuous at $(t_1,x_1) = (T,x_1)$, there exists $\delta > 0$ such that

$$(t,x) \in \overline{D}, \quad |t-T| + |x-x_1| < \delta \quad \Rightarrow \quad |u(t,x) - u(T,x_1)| < \eta.$$

- If $0 < T - \tau < \delta$, then we have

$$m - \eta = u(T,x_1) - \eta \leq u(\tau,x_1) \leq m$$

  and then $m - \eta \leq m(\tau) \leq m$.

- This means that $\lim_{\tau \uparrow T} m(\tau) = m$. The proof has been completed. $\square$

- Now we consider the following initial-boundary value problem:

$$\begin{cases} u_t - \Delta u = f(t,x), & 0 < t < T, \quad x \in U, \\ u(t,x) = g(x), & 0 < t < T, \quad x \in \partial U, \\ u(0,x) = u_0(x), & x \in U \end{cases} \tag{3.5}$$

where $U \subset \mathbb{R}^N$ is a bounded domain, $f \in C((0,T) \times U)$, $g \in C(\partial U)$ and $u_0 \in C(U)$ are given.

Corollary 3.2

Suppose that U is a bounded domain and $D = (0,T) \times U$. Let $u \in C(\overline{D}) \times C^{1,2}(D)$.

(1) If $u_t - \Delta u \leq 0$ in D, then

$$\max_{(t,x) \in \overline{D}} u(t,x) = \max_{(t,x) \in \partial_p D} u(t,x).$$

(2) If $u_t - \Delta u \geq 0$ in D, then

$$\min_{(t,x) \in \overline{D}} u(t,x) = \min_{(t,x) \in \partial_p D} u(t,x).$$

Corollary 3.3

Suppose that U is a bounded domain. The initial-boundary value problem (3.5) has at most one solution in $C(\overline{D}) \cap C^{1,2}(D)$, where $D = (0,T) \times U$.

Proposition 3.4(Comparison Principle)

Let U be a bounded domain and $D = (0,T) \times U$ and let $u_1,\ u_2 \in C(\overline{D}) \cap C^{1,2}(D)$ be the solution to initial-boundary value problem (3.5) with $f = f_i$, $g = g_i$ and $u_0 = u_{0,i}$ $(i = 1,2)$. If $f_1 \geq f_2$, $g_1 \geq g_2$ and $u_{0,1} \geq u_{0,2}$ then $u_1 \geq u_2$ in D.

- Now we consider the case where $U = \mathbb{R}^N$.

Theorem 3.5

Let $D = (0,T) \times \mathbb{R}^N$ and $u \in C(\overline{D}) \cap C^{1,2}(D)$ be a solution of (3.1) with initial condition $u(0,x) = u_0(x)$ and suppose u_0 is bounded on $\mathbb{R}^N$. If there exists positive constants M and c such that $|u(t,x)| \leq Me^{c|x|^2}$ in D, then $|u(t,x)| \leq \sup_{\xi \in \mathbb{R}^N} |u_0(\xi)|$ in D.

Proof:

- Considering $-u$, it is enough to prove that $u(t,x) \le \sup_{\xi\in\mathbb{R}^N} u_0(\xi)$ holds.
- We first assume that $4cT < 1$. We take $\varepsilon > 0$ so that $4c(T+\varepsilon) < 1$ holds. Fix any $y \in \mathbb{R}^N$.
- Consider

$$u_\theta(t,x) = u(t,x) - \theta\{4\pi(T+\varepsilon-t)\}^{-N/2} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}},$$

  where $\theta > 0$ is any small constant.
- We note that the second term of u_θ satisfies the heat equation. Thus u_θ also satisfies the heat equation.
- Now we consider $E = \{(t,x) : 0 < t < T, |x-y| < \rho\}$ for any $\rho > 0$. By Theorem 3.1 we have

$$u_\theta(t,x) \le \max_{(\tau,\xi)\in\partial_p E} u_\theta(\tau,\xi) \quad \text{for} \quad 0 \le t \le T, \ |x-y| \le \rho.$$

- On $\{(t,x) : t = 0, |x-y| < \rho\}$ we have $u_\theta(t,x) \le u(t,x) \le \sup_{\xi\in\mathbb{R}^N} u_0(\xi)$.
- On $\{(t,x) : 0 < t < T, |x-y| = \rho\}$ we have

$$\begin{aligned} u_\theta(t,x) &\le M e^{c|x|^2} - \theta\{4\pi(T+\varepsilon-t)\}^{-N/2} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \\ &\le M e^{c(|y|+\rho)^2} - \theta\{4\pi(T+\varepsilon-t)\}^{-N/2} e^{\frac{\rho^2}{4(T+\varepsilon-t)}} \le \sup_{\xi\in\mathbb{R}^N} u_0(\xi) \end{aligned}$$

  for sufficiently large ρ since $4cT < 1$. Therefore $\max_{(\tau,\xi)\in\partial_p E} u_\theta(\tau,\xi) \le \sup_{\xi\in\mathbb{R}^N} u_0(\xi)$ and

$$u_\theta(t,y) = u(t,y) - \theta\{4\pi(T+\varepsilon-t)\}^{-N/2} \le \sup_{\xi\in\mathbb{R}^N} u_0(\xi) \quad \text{for} \quad 0 \le t \le T.$$

  By letting $\theta \to 0$ we obtain the desired inequality.
- If $4cT \ge 1$, take $l > 0$ so that $4cl < 1$ holds.
- By the above argument we have $|u(t,x)| \le \sup_{\xi\in\mathbb{R}^N} |u_0(\xi)|$ for $0 \le t \le l$, $x \in \mathbb{R}^N$. We next use the above argument by regarding $t = l$ as an initial time to obtain

$$|u(t,x)| \le \sup_{\xi\in\mathbb{R}^N} |u(l,\xi)| \le \sup_{\xi\in\mathbb{R}^N} |u_0(\xi)| \quad \text{for} \quad l \le t \le 2l, \ x \in \mathbb{R}^N$$

  and therefore

$$|u(t,x)| \le \sup_{\xi\in\mathbb{R}^N} |u_0(\xi)| \quad \text{for} \quad 0 \le t \le 2l, \quad x \in \mathbb{R}^N.$$

  holds.

- By repeating this argument N times so that $Nl > T$ holds we obtain

$$|u(t,x)| \leq \sup_{\xi \in \mathbb{R}^N} |u_0(\xi)| \quad \text{for} \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^N$$

The proof has been completed. □

- For the initial value problem:

$$\begin{cases} u_t - \Delta u = f(t,x), & 0 < t < T, \quad x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{3.6}$$

we have the following uniqueness result.

Corollary 3.6

Let $D = (0,T) \times \mathbb{R}^N$ and suppose that u_0 is a bounded function on $\mathbb{R}^N$. Then initial value problem (3.6) has at most one solution $u \in C(\overline{D}) \cap C^{1,2}(D)$ which satisfies the growth condition $|u(t,x)| \leq Me^{c|x|^2}$ for some $M > 0$ and $c > 0$.

Remark: If we do note impose any growth condition, the uniqueness result does not hold.

3.1.2 Strong maximum principle for the heat equation*

- As the harmonic functions, we can obtain the strong maximum principle via a mean value property for the heat equations.

- Let us define the fundamental solution of heat equation (3.1):

$$G(t,x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}.$$

This function is called the **heat kernel**. This function satisfies the heat equation.

- It is well known that $u(t,x)$ defined by

$$u(t,x) = \int_{\mathbb{R}^N} G(t-s, x-y) u_0(y) dy$$

satisfies

$$\begin{cases} u_t - \Delta u = 0, & t > 0, \; x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{3.7}$$

under a suitable condition on u_0. Therefore by Corollary 3.4, this u is a unique solution to (3.7) under the growth condition in the corollary.

- For fixed $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$ and $r > 0$ we define

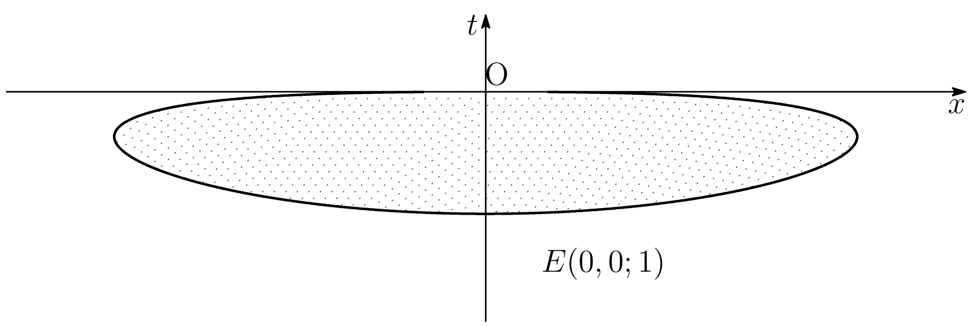
$$E(t, x; r) = \left\{ (y, s) \in \mathbb{R}^{N+1} : s \leq t, \quad G(x - y, t - s) \geq \frac{1}{r^N} \right\}.$$

  This set is called **heat ball**.

- Let us consider the case for $N = 1$, $x = 0$, $t = 0$ to illustrate the heat ball in $\mathbb{R}^2$. A point $(s, y) \in E(0, 0; r)$ if and only if (note that $s < 0$)

$$\begin{aligned}
&\frac{1}{\sqrt{4\pi(-s)}} e^{-\frac{y^2}{4(-s)}} \geq \frac{1}{r}, \\
&r e^{\frac{y^2}{4s}} \geq 2\sqrt{\pi(-s)}, \\
&\log r + \frac{y^2}{4s} \geq \frac{1}{2} \log 4\pi + \log(-s), \\
&\frac{y^2}{4s} \geq \frac{1}{2} \log \frac{4\pi(-s)}{r^2}, \\
&y^2 \leq 2s \log \frac{4\pi(-s)}{r^2}, \\
&y^2 \leq 2(-s) \log \frac{r^2}{4\pi(-s)},
\end{aligned}$$

- The following figure expresses $E(0, 0; 1)$:



- Now we give the mean value formula for the heat equation.

Theorem 3.7(Mean value propaety for the heat equation)

Let $D = (0, T) \times U$ with domain $U \subset \mathbb{R}^N$ and $u \in C^{1,2}(D)$ solve (3.1). Then for each $E(t, x; r) \subset D$ it holds that

$$u(t, x) = \frac{1}{4r^N} \iint_{E(t,x;r)} u(s, y) \frac{|x - y|^2}{(t - s)^2} ds dy.$$

- For the proof see [**Evans**].
- It holds that

$$\frac{1}{4r^N}\iint_{E(t,x;r)} \frac{|x-y|^2}{(t-s)^2}dsdy = 1.$$

Theorem 3.8(Strong maximum principle for the heat equation)

Let $D = (0,T)\times U$ with a bounded domain $U \subset \mathbb{R}^N$ and $u \in C^{1,2}(D)$ solve (3.1). If there exists $(t_0, x_0) \in D$ such that

$$u(t_0, x_0) = \max_{(t,x)\in\overline{D}} u(t,x),$$

then u is constant in $\{(t,x) : 0 \le t \le t_0, x \in \overline{U}\}$.

Proof: **Step 1**

- Let $u(t_0, x_0) = \max_{(t,x)\in\overline{D}} u(t,x) =: M$.
- For all sufficiently small $r > 0$, $E(t_0, x_0; r) \subset D$ and we use the mean-value property to obtain

$$M = u(t_0, x_0) = \frac{1}{4r^N}\iint_{E(t_0,x_0;r)} u(y,s)\frac{|x_0-y|^2}{(t_0-s)}dsdy \le M$$

  and $u \equiv M$ in $E(t_0, x_0; r)$ since

$$\frac{1}{4r^N}\iint_{E(t,x;r)} \frac{|x-y|^2}{(t-s)^2}dsdy = 1.$$

- Now we draw any line segment L in D connecting (t_0, x_0) with some other point $(s_0, y_0) \in D$ with $0 \le s_0 < t_0$. Define

$$r_0 = \inf\{s \ge s_0 : u(t,x) = M \ \ ^\forall(t,x) \in L, s \le t \le t_0\}.$$

- Since $u \equiv M$ in $E(t_0, x_0; r)$, $r_0 < t_0$. Moreover since u is continuous, the above minimum is attained. Assume $r_0 > s_0$. Then tere exists $(r_0, z_0) \in L \cap D$ such that $u(r_0, z_0) = M$. Taking $r > 0$ sucfficiently small so that $E(r_0, z_0; r) \subset D$ we obtain $u \equiv M$ in $E(r_0, z_0; r)$.
- Since $E(r_0, z_0; r)$ contains $L \cap \{r_0 - \sigma \le t \le r_0\}$ for some small $\sigma > 0$, we get a contradiction. Therefore $r_0 = s_0$ and thus $u \equiv M$ on L.

Step 2

- Take any point $(t, x) \in D$. Since U is connected there exists a polygonal arc which connects (t_0, x_0) with (t, x): there exists points $x_0, x_1, \cdots, x_m = x$ in U such that each segment connects x_i with x_{i+1} $(i = 1, \ldots, m)$. Choose times $t_0 > t_1 > \cdots > t_m = t$.
- By the argument in Step 1 we obtain: $u(t_0, x_0) = u(t_1, x_1) = \cdots = u(t_m, x_m) = u(t, x)$. The proof has been completed. $\Box$

3.2 Maximum principles for general parabolic equations

- Now we introduce the following operator

$$\mathcal{L}u = -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(t,x)u_{x_i x_j} + \sum_{i=1}^{N} b_i(t,x)u_{x_i} + c(t,x)u$$

for given coefficients a_{ij}, b_i, c. For simplicity we assume that these coefficients are continuous and bounded functions.

Definition

Let $D \subset \mathbb{R}^{N+1}$ be a domain. The operator $\frac{\partial}{\partial t} + L$ is called **uniformly parabolic** if there exists a constant $\theta > 0$ such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(t,x)\xi_i\xi_j \geq \theta|\xi|^2$$

for $(t, x) \in D$ and $\xi = {}^t(\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$.

- We define

$$\mathcal{P} := \frac{\partial}{\partial t} + \mathcal{L}.$$

3.2.1 Weak maximum principle

Lemma 3.9

Let $D \subset \mathbb{R}^{N+1}$ be a domain and $\mathcal{P}$ be uniformly parabolic and $c(t, x) \geq 0$ in D. Assume $u \in C(\overline{D}) \cap C^{1,2}(D)$ satisfies $\mathcal{P}u < 0$. If u has a nonnegative maximum over $\overline{D}$, then u cannot attain this maximum at any point in D.

Proof:

- Suppose that u takes it nonnegative maximum at $(t_0, x_0) \in D$.

- By the same argument as in the proof of Lemma 2.1 we can see

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(t_0,x_0)u_{x_ix_j}(t_0,x_0) \geq 0.$$

- Since $(t_0,x_0) \in D$ which is interior point of D, we also have $u_{x_i}(t_0,x_0) = 0$ and $u_t(t_0,x_0) = 0$. Therefore

$$\begin{aligned}\mathcal{P}u(t_0,x_0) =& u_t(t_0,x_0) - \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(t_0,x_0)u_{x_ix_j}(t_0,x_0) \\ &\qquad + \sum_{i=1}^{N} b_i(t_0,x_0)u_{x_i}(t_0,x_0) + c(t_0,x_0)u(t_0,x_0) \geq 0,\end{aligned}$$

  which is a contradiction to $\mathcal{P}u < 0$. □

Remark:

(1) If $c(t,x) \equiv 0$, then the requirement for nonnegativeness of the maximum can be removed.

(2) When $D = (0,T) \times U$ with a domain U and if $u(t_0,T) = \max_{(t,x)\in\overline{D}} u(t,x)$, then $u_t(T,x_0) \geq 0$ and $u_{x_i}(T,x_0) = 0$. Hence we obtain a contradiction. Therefore u cannot attain the maximum at any point in $\overline{D} \setminus \partial_p D$.

Theorem 3.10(Weak maximum principle)

Let $U \subset \mathbb{R}^N$ be a bounded domain, $T > 0$ and $D = (0,T) \times U$. Suppose that $\mathcal{P}$ is uniformly parabolic and $c(t,x) \geq 0$ in D. Assume $u \in C(\overline{D}) \cap C^{1,2}(D)$ satisfies $\mathcal{P}u \leq 0$ and $M = \max_{(t,x)\in\overline{D}} u(t,x) \geq 0$. Then it holds that

$$\max_{(t,x)\in\overline{D}} u(t,x) = \max_{(t,x)\in\partial_p D} u(t,x)$$

Proof:

- It suffices to show that

$$\max_{(t,x)\in\overline{D}} u(t,x) \leq \max_{(t,x)\in\partial_p D} u(t,x) \tag{3.8}$$

- Let $v(t,x) = u(t,x) + \varepsilon e^{-kt}$ for $\varepsilon > 0$. Then

$$\mathcal{P}v = \mathcal{P}u + \varepsilon\mathcal{P}e^{-kt} = \mathcal{P}u + \varepsilon(-k + c(t,x))e^{-kt} \leq \varepsilon(-k + c(t,x))e^{-kt} < 0$$

  for sufficiently large $k > 0$ since c is a bounded function.

- Moreover $\max_{(t,x)\in\overline{D}} v(t,x) > M \geq 0$. Hence by Lemma 3.9 and its Remark-(2),

$$\max_{(t,x)\in\overline{D}} u(t,x) \leq \max_{(t,x)\in\overline{D}} v(t,x) = \max_{(t,x)\in\partial_p D} v(t,x) \leq \max_{(t,x)\in\partial_p D} u(t,x) + \varepsilon.$$

- By letting $\varepsilon \to 0$ we obtain (3.8). $\square$

Remark: If $c(t,x) \equiv 0$, then the requirement for nonnegativeness of M can be removed.

3.2.2 Strong maximum principle

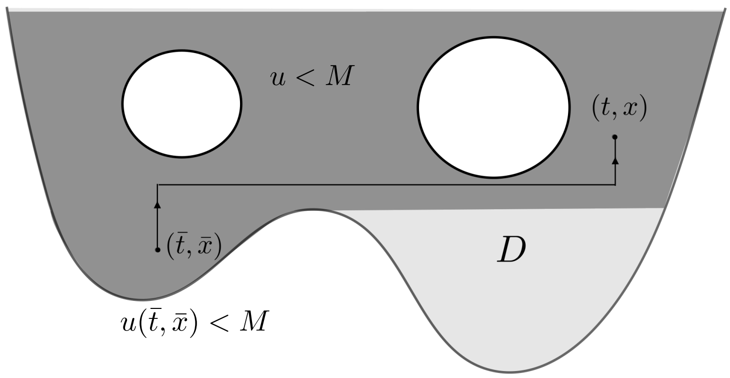
Theorem 3.11

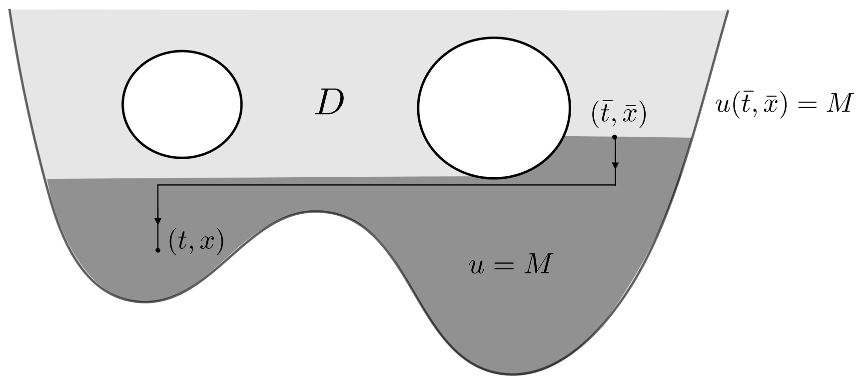
Let $D \subset \mathbb{R}^{N+1}$ be a domain and $\mathcal{P}$ be uniformly parabolic and $c(t,x) \geq 0$ (resp. ≤ 0)in D. Assume $u \in C(\overline{D}) \cap C^{1,2}(D)$ satisfies $\mathcal{P}u \leq 0$ (resp. ≥ 0) and $M := \max_{(t,x)\in\overline{D}} u(t,x) \geq 0$ (resp. $M := \inf_{(t,x)\in\overline{D}} \leq 0$). Suppose that $u(\bar{t},\overline{x}) < M$ (resp. $> M$) for some $(\bar{t},\overline{x}) \in D$. Then $u(t,x) < M$ (resp. $> M$) at all points (t,x) in D which can be connected to $(\bar{t},\overline{x})$ by an arc in D consisting of finite number of horizontal and **upward** vertical segments.

Remark: If $c(t,x) \equiv 0$, then the requirement for nonnegativeness (resp. nonpositivensess) of M can be removed.

Corollary 3.12

Suppse the same assumptions of Theorem 3.11 on D, $\mathcal{P}$ and $c(t,x) \geq 0$ in D. Let $M := \sup_{(t,x)\in\overline{D}} u(t,x) \geq 0$ and $u(\bar{t},\overline{x}) = M$ for some $(\bar{t},\overline{x}) \in D$. Then $u(t,x) = M$ at any points (t,x) which can be connected to $(\bar{t},\overline{x})$ by an arc in D consisting of finite number of horizontal and **downward** vertical segments.





- When $D = (T_1, T_2) \times U$ with a domain $U \subset \mathbb{R}^N$ we obtain the following corollary.

Corollary 3.13

Let $U \subset \mathbb{R}^N$ be a domain, $D = (T_1, T_2) \times U$, with $-\infty < T_1 < T_2 \leq \infty$, and $\mathcal{P}$ be uniformly parabolic and $c(t, x) \geq 0$ in D. Assume $u \in C(\overline{D}) \cap C^{1,2}(D)$ satisfies $\mathcal{P}u \leq 0$ and $M := \max_{(t,x)\in\overline{D}} u(t, x) \geq 0$. Suppose that $u(\overline{t}, \overline{x}) = M$ for some $(\overline{t}, \overline{x}) \in D$. Then $u(t, x) = M$ in $\{(t, x) \in D : T_1 \leq t \leq \overline{t}\}$.

- For the proof of Theorem 3.11 we need some lemmas.
- Hereafter we assume $c(t, x) \equiv 0$.

Lemma 3.14

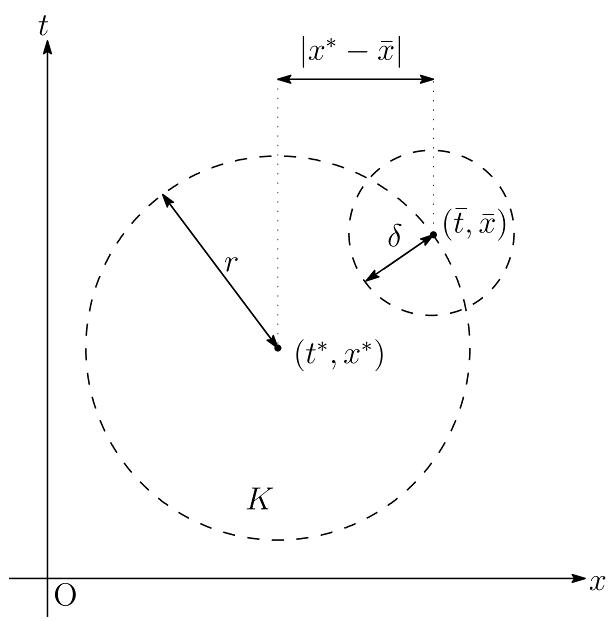
Let $D \subset \mathbb{R}^{N+1}$ be a domain and $\mathcal{P}$ be uniformly parabolic in D and $K \subset \mathbb{R}^{N+1}$ be a ball with $\overline{K} \subset D$. Assume $u \in C(\overline{D}) \cap C^{1,2}(D)$ satisfies $\mathcal{P}u \leq 0$. Let $M := \max_{(t,x)\in\overline{D}} u(t,x)$ and $u(t,x) < M$ for $(t,x) \in K$. If $u(\bar{t},\bar{x}) = M$ for some $(\bar{t},\bar{x}) \in \partial K$, then $(\bar{t},\bar{x})$ is a "north pole" or "south pole" of K. More precisely, if

$$K = B_r(t^*,x^*) = \{(t,x) \in \mathbb{R}^{N+1} : (t-t^*)^2 + |x-x^*|^2 < r^2\},$$

then $\bar{x} = x^*$ and $\bar{t} = x^* + r$ or $\bar{t} = x^* - r$.

Proof:

- Set $x^* = {}^t(x_1^*, \ldots, x_N^*)$ amd $\bar{x} = {}^t(\bar{x}_1, \ldots, \bar{x}_N)$.
- By considering smaller ball which is tangent to $B_r(t^*,x^*)$ at $(\bar{t},\bar{x})$ we may assume that $(\bar{t},\bar{x})$ is a only point in ∂K such that $u(\bar{t},\bar{x}) = M$.
- Consider $w(t,x) = e^{-\alpha(|x-x^*|^2+(t-t^*)^2)} - e^{-\alpha r^2}$ with constant $\alpha > 0$ is determined later. w satisfies $w = 0$ on ∂K, $w > 0$ in K and $w < 0$ in $\overline{K}^c$.
- Suppose that $(\bar{t},\bar{x})$ is neither the north pole of K nor the south pole of K. Then $|x^* - \bar{x}| > 0$
- Take $\delta \in (0, |x^* - \bar{x}|)$ so that $B_\delta(\bar{t},\bar{x}) \subset D$ holds.



- By the direct calculation we obtain

$$\mathcal{P}w = \Bigg\{ -2\alpha(t-t^*) - \sum_{i,j=1}^{N} 4a_{ij}\alpha^2(x_i - x_i^*)(x_j - x_j^*) + \sum_{i=1}^{N} \left(2a_{ii}\alpha - 2b_i\alpha(x_i - x_i^*)\right) \Bigg\} e^{-\alpha(|x-x^*|^2+(t-t^*)^2)}.$$

- Since

$$\sum_{i,j=1}^{N} a_{ij}(x_i - x_i^*)(x_j - x_j^*) \geq \theta |x - x^*|^2$$

and a_{ii} and b_i are bounded in D, there exists a constants $C_1, C_2 > 0$ such that

$$\mathcal{P}w \leq \left\{ -2\alpha(t-t^*) - 4\alpha^2\theta|x-x^*|^2 + \alpha C_1 + C_2\alpha|x-x^*| \right\} e^{-\alpha(|x-x^*|^2+(t-t_0)^2)}.$$

- On $\overline{B_\delta(\bar{t},\bar{x})}$ we have

$$\begin{aligned} &|x-x^*| = |x^* - x| = |(x^*-\bar{x}) - (x-\bar{x})| \geq |x^* - \bar{x}| - |x - \bar{x}| \geq r - \delta > 0, \\ &|x - x^*| \leq |x - \overline{x}| + |\overline{x} - x^*| \leq r + \delta. \end{aligned}$$

Hence

$$\mathcal{P}w \leq \left\{ 2\alpha(r+\delta) - 4\alpha^2(r-\delta)^2 + C_1\alpha + C_2\alpha(r+\delta)^2 \right\} e^{-\alpha(|x-x^*|^2+(t-t^*)^2)} < 0 \quad \text{in} \quad B_\delta(\bar{t},\bar{x})$$

for sufficiently large $\alpha > 0$.

- Let $v(t,x) = u(t,x) + \varepsilon w(t,x)$ for $\varepsilon > 0$.
- Since $u < M$ on compact set $\partial B_\delta(\bar{t},\bar{x}) \cap \overline{K}$, $v \leq M$ in $\partial B_\delta(\bar{t},\bar{x}) \cap \overline{K}$ for small $\varepsilon > 0$.
- Since $w < 0$ on $\partial B_\delta(\bar{t},\bar{x}) \cap \overline{K}^c$ and $u \leq M$ on D, $v < M$ on $\partial B_\delta(\bar{t},\bar{x}) \cap \overline{K}^c$.
- Hence $v < M$ on $\partial B_\delta(\bar{t},\bar{x})$ and $\mathcal{P}v < 0$ in $B_\delta(\bar{t},\overline{x})$.
- However since $v(\bar{t},\bar{x}) = u(\bar{t},\bar{x}) = M$ and v must achieve its maximum an interior point of $B_\delta(\bar{t},\bar{x})$, which contradicts Lemma 3.9 with $c \equiv 0$. The proof has been completed. □

Remark: When $c(t,x) \geq 0$. The above lemma is still valid if $M \geq 0$. In fact for sufficiently large $\alpha > 0$

$$\mathcal{P}w \leq \left\{ 2\alpha r - 4\alpha^2(r-\delta)^2 + C_1 + C_2\alpha(r-\delta)^2 + \|c\|_{L^\infty} \right\} e^{-\alpha(|x-x^*|^2+(t-t_0)^2)} - c(t,x)e^{-\alpha r^2} < 0 \quad \text{in} \quad B_\delta(\bar{t},\bar{x})$$

and use Lemma 3.9.

Lemma 3.15

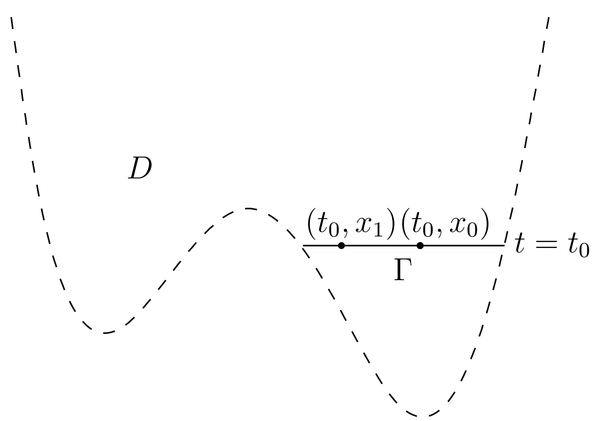
Let $D \subset \mathbb{R}^{N+1}$ be a domain and $\mathcal{P}$ is uniformly parabolic in D. Assume $u \in C^{1,2}(D)$ satisfies $\mathcal{P}u \le 0$. Let $u \le M$ in D for some $M \in \mathbb{R}$ and $u(t_0, x_0) < M$ for some $(t_0, x_0) \in D$. Then $u(t, x) < M$ holds in the component of $\{(t, x) \in D : t = t_0\}$ which contains (t_0, x_0).

Proof: For simplicity we give the proof only for the case where $N = 1$.

- Let Γ is the component of $\{(t, x) \in D : t = t_0\}$ which contains (t_0, x_0).

- Suppose that $u(t_0, x_1) = M$ for some $(t_0, x_1) \in \Gamma$ and we will get a contradiction. We may assume that

$$u(t, x) < M \quad \text{for} \quad |x - x_0| < |x_0 - x_1|.$$



- Let $d_0 = \min\{|x_1 - x_0|, \mathrm{dist}(x_1, \partial D)\}$.

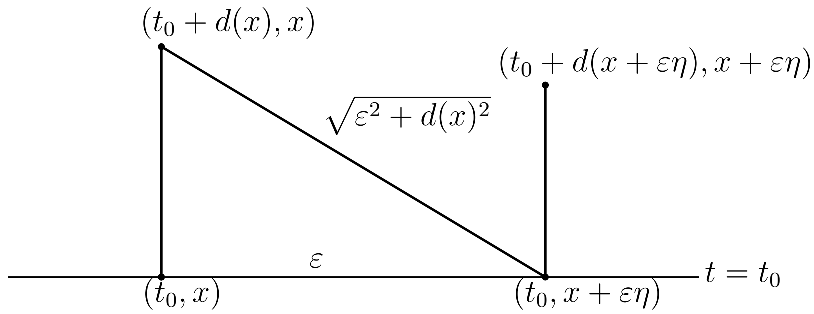
- For $0 < |x - x_1| < d_0$ we define $d(x)$ as the distance from (t_0, x) to the nearest point in D where $u = M$. Since $u(t_0, x_1) = M$, $d(x) \le |x - x_1|$.

- Since u is continuous and $u(t_0, x) < M$ for $0 < |x - x_1| < d_0$, for any $(t_0, x) \in D$ with $0 < |x - x_1| < d_0$ there exists a ball centered at (t_0, x) in which $u < M$. Hence $d(x) > 0$.

- Applying Lemma 3.14 we obtain for any $(t_0, x) \in D$ with $0 < |x - x_1| < d_0$, $u(t_0 + d(x), x) = M$ or $u(t_0 - d(x), x) = M$.

- Let $\varepsilon > 0$ be a small number and $|\eta| = 1$. By the definition of $d(x)$ we have

$$d(x + \varepsilon\eta) \le \sqrt{\varepsilon^2 + d(x)^2} < \sqrt{\left(d(x) + \frac{\varepsilon^2}{2d(x)}\right)^2} = d(x) + \frac{\varepsilon^2}{2d(x)}. \tag{3.9}$$



- By exchanging x and $x+\varepsilon\eta$ we obtain $d(x) \le \sqrt{\varepsilon^2 + d(x+\varepsilon\eta)^2}$ and also

$$d(x+\varepsilon\eta) \ge \sqrt{d(x)^2 - \varepsilon^2}. \tag{3.10}$$

- Let us show $d(x) \equiv 0$ for $0 < |x - x_1| < \delta_0$. If this were shown, then $u(t_0, x) \equiv M$ for $|x - x_1| < \delta_0$ which is a contradiction.

- Take x with $0 < |x - x_1| < \delta_0$ and $\varepsilon \in (0, d(x))$. We subdivide the interval $(x, x+\varepsilon)$ (or $(x-\varepsilon, x)$) into k equal parts and apply (3.9) and (3.10) to get

$$\begin{aligned} & d\left(x + \frac{j+1}{k}\varepsilon\eta\right) - d\left(x + \frac{j}{k}\varepsilon\eta\right) \\ \le\ & d\left(x + \frac{j}{k}\varepsilon\eta\right) + \frac{\varepsilon^2}{2k^2 d(x + (j/k)\varepsilon\eta)} - d\left(x + \frac{j}{k}\varepsilon\eta\right) \\ \le\ & \frac{\varepsilon^2}{2k^2\sqrt{d(x)^2 - (j/k)^2\varepsilon^2}} \le \frac{\varepsilon^2}{2k^2\sqrt{d(x)^2 - \varepsilon^2}} \end{aligned}$$

  for $j = 0, \cdots, k-1$.

- By summing the above inequalities from $j = 0$ to $j = k-1$ we obtain

$$d(x+\varepsilon\eta) - d(x) \le \frac{\varepsilon^2}{2k\sqrt{d(x)^2 - \varepsilon^2}}.$$

- Letting $k \to \infty$ we obtain $d(x+\varepsilon\eta) \le d(x)$ for each small $\varepsilon > 0$ and $|\eta| = 1$. Since $0 \le d(x) \le |x - x_1|$, we can conclude that $d(x) \equiv 0$. $\square$

Lemma 3.16

Let $D \subset \mathbb{R}^{N+1}$ be a domain and $\mathcal{P}$ is uniformly parabolic in D. Assume $u \in C^{1,2}(D)$ satisfies $\mathcal{P}u \le 0$ and

$$u < M \quad \text{in} \quad \{(t,x) \in D : t_0 < t < t_1\},$$

for some $t_0 < t_1$. Then

$$u < M \quad \text{on} \quad \{(t,x) \in D : t = t_1\}.$$

Proof:

- Suppose that $u(t_1, x^*) = M$. Take $r > 0$ so that

$$B_r(t_1, x^*) \subset \{(t, x) \in D : t > t_0\}.$$

- Define

$$w(t, x) := e^{-|x-x^*|^2-\alpha(t-t_1)} - 1,$$

where $\alpha > 0$ is choosen later.

- A direct calculation implies that

$$\begin{aligned}\mathcal{P}w &= e^{-|x-x^*|^2-\alpha(t-t_1)}\times \\ &\qquad \left\{-\alpha - 4\sum_{i,j=1}^{N} a_{ij}(x_i - x_i^*)(x_j - x_j^*) + 2\sum_{j=1}^{N}(a_{ii} + b_i(x_i - x_i^*))\right\} \\ &\le e^{-|x-x^*|^2-\alpha(t-t_1)}\times \\ &\qquad \left\{-\alpha - 4\theta|x - x^*|^2 + 2\sum_{i=1}^{N}(a_{ii} + b_i(x_i - x_i^*))\right\}\end{aligned}$$

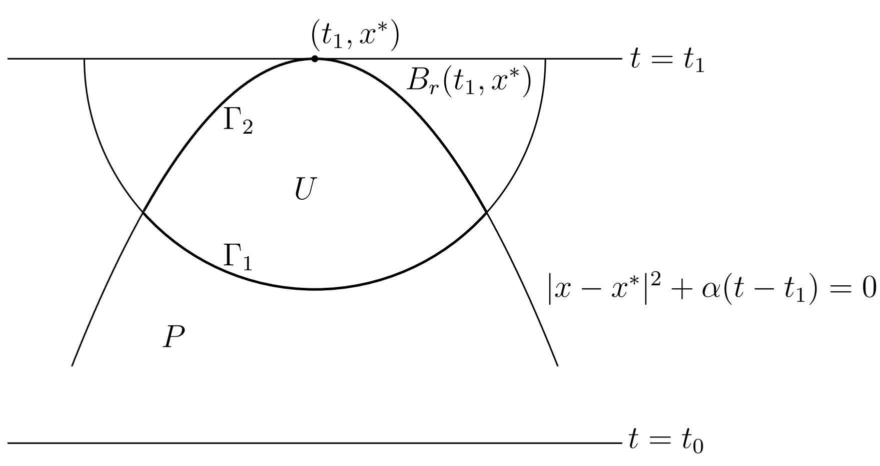
Thus we can choose $\alpha > 0$ so that $\mathcal{P}w < 0$ in $B_r(t_1, x^*)$.

- Next we consider the paraboloid

$$|x - x^*|^2 + \alpha(t - t_1) = 0,$$

which is tangent to the hyperplane $\{(t, x) : t = t_1\}$ at (t_1, x^*).

- Let $P = \{(t, x) \in D : \alpha(t - t_1) < -|x - x^*|^2\}$, $\Gamma_1 = \partial B_r(t_1, x^*) \cap P$, $\Gamma_2 = \partial P \cap B_r(t_1, x^*)$ and let U denote the domain determined by Γ_1 and Γ_2.



- Let $v(t,x) = u(t,x) + \varepsilon w(t,x)$ for $\varepsilon > 0$. Then we have $\mathcal{P}v < 0$ in $B_r(t_1, x^*)$.
- Since $u < M$ on compact set Γ_1 there exists $\delta > 0$ such that $u \le M - \delta$ on Γ_1, thus for sufficiently small $\varepsilon > 0$ it holds that $v \le M$ on Γ_2.
- On Γ_2 it holds that $u \le M$ since $w = 0$ on Γ_2. Moreover $v = M$ at $(t_1, x^*) \in \Gamma_2$. Therefore $v \le M$ on ∂U.
- By $\mathcal{P}v < 0$ in $B_r(t_1, x^*)$ and Lemma 3.9, v cannot attain its maximum over $\overline{U}$ at any point in U.
- Thus v attains its maximum over $\overline{U}$ at a point on ∂U. Therefore M is the maximum of v and it is attained at (t_1, x^*).
- Hence at (t_1, x^*) we obtain $\dfrac{\partial v}{\partial t} \ge 0$. But $\dfrac{\partial w}{\partial t} = -\alpha < 0$, therefore $\dfrac{\partial u}{\partial t} > 0$ at (t_1, x^*).
- Since u attaines its maximum at (t_1, x^*) we have

$$u_{x_i}(t_1, x^*) = 0 \quad i = 1, \cdots, N,$$
$$\sum_{i,j=1}^{N} a_{ij}(t_1, x^*) u_{x_i x_j}(t_1, x^*) \le 0.$$

and then $\mathcal{P}u > 0$ at (t_1, x^*). However this is a contradiction to $\mathcal{P}u \le 0$ in D. $\square$

Remark: When $c(t,x) \ge 0$ in D, Lemma 3.16 is still valid if $M \ge 0$. In fact when $c(t,x) \ge 0$ and α is sufficiently large

$$\mathcal{P}w \le e^{-|x-x^*|^2 - \alpha(t_1 - t)} \times \left\{ -\alpha - 4\theta|x - x^*|^2 + 2\sum_{j=1}^{N} (a_{ii} + b_i(x_i - x_i^*)) \right\} - c(t,x) \le 0.$$

Therefore we can use Lemma 3.9.

Proof of Theorem 3.11:
Step 1: If $u(\bar{t}, \overline{x}) < M$ then

$$u < M \quad \text{on} \quad l = \{(t, \overline{x}) : \bar{t} \le t \le t_1\} \subset D.$$

- Let $\tau = \sup\{t \ge \bar{t} : u(\cdot, \overline{x}) < M \text{ on } [\bar{t}, t]\}$.
- Suppose that $\tau \le t_1$. By continuity $u(\tau, \overline{x}) = M$.
- By Lemma 3.14 $u < M$ on the component of $\{(t,x) \in D : \bar{t} \le t < \tau\}$ which contains $(\bar{t}, \overline{x})$.

- By Lemma 3.16, $u(\tau, \overline{x}) < M$ which is a contradiction.

Step 2: Completion of the proof of Theorem 3.11.

- Let $\overline{\mathrm{P}}(\bar{t}, \overline{x}) \in D$ be a point such that $u(\bar{t}, \overline{x}) < M$ and $\mathrm{P}(t, x) \in D$ be any point which can be connected to $\overline{P}$ by an arc in D consisting of a finite number of hrizontal and vertical upward segments.

- Hence there are points $\overline{\mathrm{P}} = \mathrm{P}_0, \mathrm{P}_1, \ldots, \mathrm{P}_k = \mathrm{P}$ in D where P_i is connected to P_{i+1} by either a horizontal or upward vertical segment contained in D.

- By Lemmas 3.14 and Step 1 we obtain $u(t, x) < M$. The proof has been completed. $\square$

- Next theorem is the parabolic version of the Hopf lemma.

Theorem 3.17

Let $D \subset \mathbb{R}^{N+1}$ be a bounded domain and $\mathcal{P}$ be uniformly parabolic in D. Assume $u \in C^{1,2}(D) \cap C(\overline{D})$ satisfies $\mathcal{P}u \leq 0$. Suppose

- $\max_{(t,x) \in \overline{D}} u(t, x) = M (\geq 0)$ is attained at $p = (t_0, x_0) \in \partial D$,

- D satisfies the interior sphere condition at p, that is, there is a ball $B_r(p_1) \subset D$ with $\partial D \cap \partial B_r(p_1) = \{p\}$,

- $u < M$ in D,

- the radial direction from p_1 to p is not parallel to the t-axis.

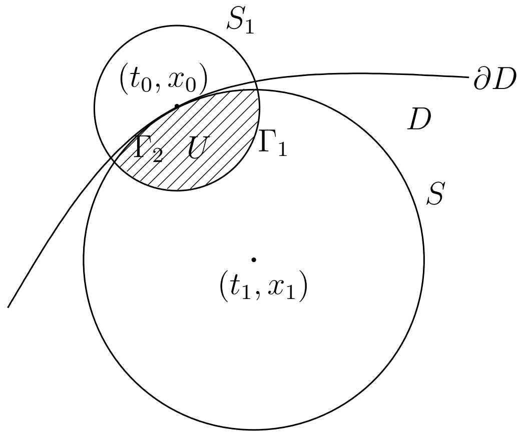
Then $\dfrac{\partial u}{\partial \nu}(t_0, x_0) > 0$ for every outward direction ν.

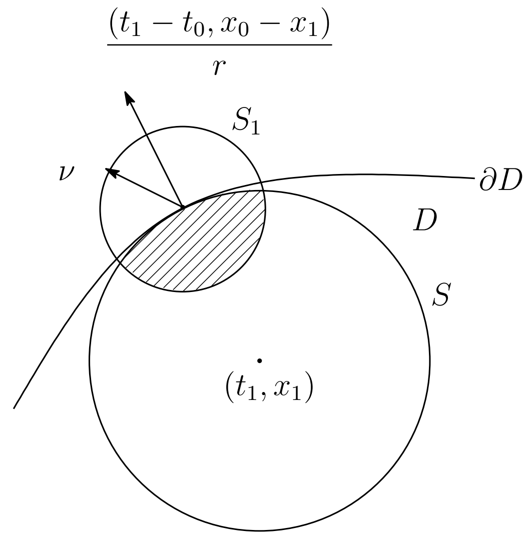
Proof:

- $S = B_r(t_1, x_1)$.

- By the assumption $|x_1 - x_0| > 0$. Take $0 < \rho < |x_1 - x_0|$ and conider $S_1 = B_\rho(t_0, x_0)$.

- Let $\Gamma_1 = \partial S_1 \cap \overline{S}$, $\Gamma_2 = \partial S \cap S_1$ and U be the region enclosed by Γ_1 and Γ_2.





- Since $u < M$ on compact set Γ_1, there exists $\delta > 0$ such that $u \le M - \delta$ on Γ_1.
- We also have $u < M$ on $\Gamma_2 \setminus \{(t_0, x_0)\}$ and $u(t_0, x_0) = M$.
- Let $w(t, x)$ be the auxiliary function defined by

  $$w(t, x) = e^{-\alpha\{|x - x_1|^2 + (t - t_1)^2\}} - e^{-\alpha r^2}$$

  for $\alpha > 0$.
- Cleary $w = 0$ on ∂S.
- By the same computation as in the proof of Lemma 3.14 (and its remark) we obtain $\mathcal{P}w < 0$ in S_1 for sufficiently large $\alpha > 0$.
- Let $v = u + \varepsilon w$ with $\varepsilon > 0$. Then $\mathcal{P}v < 0$ in D.
- Since $u \le M - \delta$ on Γ_1, there exists $\varepsilon > 0$ such that $v < M$ on Γ_1.
- Since $w = 0$ on ∂S we have $v < M$ on $\Gamma_2 \setminus \{(t_0, x_0)\}$ and $v(t_0, x_0) = M$.
- Hence by Lemma 3.9, v cannot have nonnegative maximum over $\overline{U}$ in any point U and therefore the maximum of v over $\overline{U}$ attains at only (t_0, x_0).
- Hence at (t_0, x_0) we obtain

  $$\frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} + \varepsilon \frac{\partial w}{\partial \nu} \ge 0$$

  for any outward direction ν.

- But

$$D_{(t,x)}w(t_0,x_0) := (u_t(t_0,x_0), D_x u(t_0,x_0))$$
$$= -2\alpha(t_0-t_1, x_0-x_1)e^{-\alpha\{|x-x_1|^2+(t-t_1)^2\}}$$

and

$$\frac{\partial w}{\partial \nu} = -2\alpha(t_0-t_1, x_0-x_1)\cdot \nu e^{-\alpha\{|x-x_1|^2+(t-t_1)^2\}} < 0$$

since ν is outward direction.

- Therefore we obtain

$$\frac{\partial u}{\partial \nu} \geq -\varepsilon \frac{\partial w}{\partial \nu} \geq 2\varepsilon\alpha(t_0-t_1, x_0-x_1)\cdot \nu e^{-\alpha\{|x-x_1|^2+(t-t_1)^2\}} > 0$$

at (t_0, x_0). □

3.2.3 The Phragmèn-Lindelöf Principle

Theorem 3.18(Phragmèn-Lindelöf Principle)

Let $U \subset \mathbb{R}^N$ be an unbounded domain, $T > 0$ and $D = (0,T)\times U$ and suppose that $\mathcal{P}$ is uniformly parabolic in D and $u \in C^{1,2}(D)\cap C(\overline{D})$ satisfies $\mathcal{P}u \leq 0$. Assume that there exists $c > 0$ sucht that

$$\liminf_{R\to\infty} e^{-cR^2}\left(\max_{|x|=R, 0\leq t\leq T, x\in U} u(t,x)\right) \leq 0 \tag{3.11}$$

and $u(t,x) \leq 0$ in $\partial_p D$. Then $u(t,x)\leq 0$ in D.

Proof:

- Let $\rho(t,x) = e^{c\gamma|x|^2/(\gamma-ct)+\beta t}$, where $|x| = \sqrt{x_1^2+\cdots+x_N^2}$ for $x = {}^t(x_1,\ldots,x_N)$, c is the constant in (3.11) and β, γ are constant to be determined.

- Define $w(t,x) = u(t,x)/\rho(t,x)$. Since $\mathcal{P}u \leq 0$ in D we have

$$\mathcal{P}u = \rho_t w + \rho w_t - \sum_{i,j=1}^N a_{ij}\left(\rho_{x_i}w + \rho w_{x_i}\right)_{x_j} + \sum_{i=1}^N b_i\left(\rho_{x_i}w + \rho w_{x_i}\right) + c\rho h \leq 0$$

Direct caluculation implies that

$$\begin{aligned}
\mathcal{P}u &= \rho_t w + \rho w_t - \sum_{i=1}^N \sum_{j=1}^N a_{ij}\left(\rho_{x_i} w + \rho w_{x_i}\right)_{x_j} + \sum_{i=1}^N b_i\left(\rho_{x_i} w + \rho w_{x_i}\right) + c\rho \\
&= \rho w_t - \rho \sum_{i=1}^N \sum_{j=1}^N a_{ij} w_{x_i x_j} - \sum_{i=1}^N \sum_{j=1}^N (a_{ij}\rho_{x_i} w_{x_j} + a_{ij}\rho_{x_j} w_{x_i}) \\
&\qquad + \rho \sum_{i=1}^N b_i w_{x_i} + \left(-\sum_{i=1}^N \sum_{j=1}^N a_{ij}\rho_{x_i x_j} + \rho_t + \sum_{i=1}^N b_i \rho_{x_i} + c\right) w \\
&= \rho w_t - \rho \sum_{i=1}^N \sum_{j=1}^N a_{ij} w_{x_i x_j} + \sum_{i=1}^N \left(\rho b_i - \sum_{j=1}^N 2a_{ij}\rho_{x_j}\right) w_{x_i} \\
&\qquad + \rho \sum_{i=1}^N b_i w_{x_i} + \left(-\sum_{i=1}^N \sum_{j=1}^N a_{ij}\rho_{x_i x_j} + \rho_t + \sum_{i=1}^N b_i \rho_{x_i} + c\right) w
\end{aligned}$$

Since $\rho > 0$ we obtain $\tilde{\mathcal{P}}w \leq 0$, where

$$\begin{aligned}
\tilde{\mathcal{P}}w &:= w_t - \sum_{i,j=1}^N a_{ij} w_{x_i x_j} + \sum_{i=1}^N \left(b_i - \sum_{j=1}^N \frac{2a_{ij}\rho_{x_j}}{\rho}\right) w_{x_i} + \tilde{c}(t,x) w \\
&= w_t - \sum_{i,j=1}^N a_{ij} w_{x_i x_j} + \sum_{i=1}^N \left(b_i(t,x) - \sum_{j=1}^N \frac{4c\gamma x_j a_{ij}}{\gamma - ct}\right) w_{x_i} + \tilde{c}(t,x) w \\
&= w_t - \sum_{i,j=1}^N a_{ij} w_{x_i x_j} + \sum_{i=1}^N \tilde{b}_i(t,x) w_{x_i} + \tilde{c}(t,x) w, \\
\tilde{b}_i(t,x) &= b_i(t,x) - \sum_{j=1}^N \frac{4c\gamma x_j a_{ij}}{\gamma - ct}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{c}(t,x) &:= \frac{\rho_t}{\rho} - \sum_{i,j=1}^N a_{ij}\frac{\rho_{x_i x_j}}{\rho} + \sum_{i=1}^N b_i \frac{\rho_{x_i}}{\rho} + c(t,x) \\
&= \beta + \frac{c^2\gamma |x|^2}{(\gamma - ct)^2} - \sum_{i=1}^N a_{ii}\frac{2c\gamma}{\gamma - ct} - \sum_{i,j=1}^N a_{ij}\frac{4c^2\gamma^2}{(\gamma - ct)^2} x_i x_j \\
&\qquad + \sum_{i=1}^N b_i x_i \frac{2c\gamma}{\gamma - ct} + c(t,x) \\
&= \beta + \frac{c^2\gamma |x|^2}{(\gamma - ct)^2} - \frac{4c^2\gamma^2}{(\gamma - ct)^2} \sum_{i,j=1}^N a_{ij} x_i x_j + \frac{2c\gamma}{\gamma - ct} \sum_{i=1}^N (-a_{ii} + b_i x_i) + c(t,x)
\end{aligned}$$

- To use maximum principle, let us obtain some estimates for coefficients of $\tilde{\mathcal{L}}$. Since a_{ij} are bounded functions, there exists $M > 0$ such that $|a_{ij}| \leq M$ for $i, j = 1, \ldots, N$ and $(t, x) \in D$. Thus

$$|\tilde{b}_i(t,x)| \leq \|b_i\|_{L^\infty} + \frac{4c\gamma}{\gamma - ct} NM|x| \tag{3.12}$$

and

$$\begin{aligned} |\tilde{c}(t,x)| \leq \beta + \frac{c^2\gamma|x|^2}{(\gamma-ct)^2} + \frac{4c^2\gamma^2}{(\gamma-ct)^2} NM|x|^2 \\ + \frac{2c\gamma}{\gamma-ct}\left(NM + N\max_{i=1,\ldots,N}\|b_i\|_{L^\infty}|x|\right) + \|c\|_{L^\infty}. \end{aligned} \tag{3.13}$$

- Moreover we see if $\gamma - ct > 0$

$$\begin{aligned} \tilde{c}(t,x) &\geq \beta + \frac{c^2\gamma|x|^2}{(\gamma-ct)^2} - \frac{4c^2\gamma^2 NM|x|^2}{(\gamma-ct)^2} - \frac{2c\gamma NM}{\gamma-ct} \\ &\qquad - \frac{2c\gamma}{\gamma-ct} N \max_{i=1,\ldots,N}\|b_i\|_{L^\infty}|x| - \|c\|_{L^\infty} \\ &\geq \beta + \frac{c^2\gamma|x|^2}{(\gamma-ct)^2}(1 - 4\gamma NM) - \frac{2c\gamma NM}{\gamma-ct} \\ &\qquad - \frac{2c^2\gamma^2|x|^2}{(\gamma-ct)^2} - \frac{1}{2}N^2\left(\max_{i=1,\ldots,N}\|b_i\|_{L^\infty}\right)^2 - \|c\|_{L^\infty} \\ &= \beta + \frac{c^2\gamma|x|^2}{(\gamma-ct)^2}(1 - 4\gamma NM - 2\gamma) - \frac{2c\gamma NM}{\gamma-ct} \\ &\qquad - \frac{1}{2}N^2\left(\max_{i=1,\ldots,N}\|b_i\|_{L^\infty}\right)^2 - \|c\|_{L^\infty} \end{aligned} \tag{3.14}$$

- For $R > 0$ let us considerthe region

$$D_{\gamma/2c,R} = \left(0, \frac{\gamma}{2c}\right) \times \{x \in U : |x| < R\}.$$

- For $(t,x) \in D_{\gamma/2c,R}$, $\gamma/2 < \gamma - ct < \gamma$ and $|x| < R$. By (3.12) and (3.13) we see that $\tilde{b}_i$ and $\tilde{c}$ are bounded in $D_{\gamma/2c,R}$.

- We next choose $\gamma > 0$ small so that $1 - 4\gamma NM - 2\gamma > 0$. Then, by (3.14), we obtain

$$\begin{aligned} \tilde{c}(t,x) &\geq \beta - \frac{2c\gamma NM}{\gamma/2} - \frac{1}{2}N^2\left(\max_{i=1,\ldots,N}\|b_i\|_{L^\infty}\right)^2 - \|c\|_{L^\infty} \\ &= \beta - 4cNM - \frac{1}{2}N^2\left(\max_{i=1,\ldots,N}\|b_i\|_{L^\infty}\right)^2 - \|c\|_{L^\infty}. \end{aligned}$$

  Therefore we can choose $\beta > 0$ large enough so that $\tilde{c}(t,x) \geq 0$ in $D_{\gamma/2c,R}$. Note that we can choose γ and β independently R.

- Fix any $(s, y) \in (0, \gamma/2c) \times U$ and $\varepsilon > 0$. By (3.11) there exists $R_n \to \infty$ such that

$$e^{-cR_n^2} \left(\max_{|x|=R_n,\ 0 \le t \le T, x \in U} u(t,x) \right) < \varepsilon.$$

  Since $\rho(t,x)^{-1} \le e^{-cR_n^2}$ when $|x| = R_n$, we have $w < \varepsilon$ on $\partial_p D_{\gamma/2c, R_n}$ for any $n \in \mathbb{N}$.

- By the maximum principle (Theorem 3.8 or Collorary 3.9), $w < \varepsilon$ in $D_{\gamma/2c, R_n}$ for any n.

- Letting $n \to \infty$ and $\varepsilon \to 0$ we obtain $w(s, y) \le 0$ and then $u(s, y) \le 0$. In particular we obtain $u(\gamma/2c, y) \le 0$ on U.

- We can repeat the above argument with $t = \gamma/2c$ as the initial time to obtain $u \le 0$ in $(\gamma/2c, 2(\gamma/2c)) \times U$. In a finite number of steps we arrive at the conclusion. □