Advanced Topic in Modern Mathematical Sciences
Lecture Note
Maximum Principles for the Elliptic and Parabolic Partial Differential Equations

Hiroshi Matsuzawa

0 Notations

- In this section, we prepare some notations.
- $\mathbb{R}^N := \{x = {}^t(x_1, \dots, x_N) : x_i \in \mathbb{R}\}$ and

$$|x| = \sqrt{\sum_{i=1}^{N} |x_i|^2} \text{ for } \ x = {}^t(x_1, \dots, x_N) \in \mathbb{R}^N$$

- Let $U \subset \mathbb{R}^N$ be an open set, $u : U \to \mathbb{R}$ and $x \in U$.

$$\frac{\partial u}{\partial x_j}(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h}$$

provided this limit exists, where $e_i = i$ th standard coordinate vector. We also write u_{x_j} instead of $\dfrac{\partial u}{\partial x_j}$. Similarly, $\dfrac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_j x_i}$, $\dfrac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} = u_{x_k x_j x_i}$ etc. are defined.

- A vector of the form $\alpha = (\alpha_1, \dots, \alpha_N)$, where each component $\alpha_i \in \mathbb{N} \cup \{0\}$ is called a **multi-index** of order $|\alpha| = \alpha_1 + \dots + \alpha_N$.
- Given multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$, we define

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u$$

- For $k \in \mathbb{N} \cup \{0\}$, $D^k u(x) = \{D^\alpha u(x) : |\alpha| = k\}$ and

$$|D^k u| = \left(\sum_{|\alpha| = k} |D^\alpha u|^2 \right)^{1/2}$$

- For special case, if $k = 1$

$$Du(x) = (u_{x_1}(x), \dots, u_{x_N}(x))$$

and if $k = 2$

$$D^2 u(x) = \begin{pmatrix} u_{x_1 x_1}(x) & \cdots & u_{x_1 x_N}(x) \\ \vdots & \ddots & \vdots \\ u_{x_N x_1}(x) & \cdots & u_{x_N x_N}(x) \end{pmatrix}.$$

- $\Delta u = \displaystyle\sum_{k=1}^{N} \frac{\partial^2 u}{\partial x_k^2}$: Laplacian

- Now we define some function spaces:

$$C(U) = \{u : U \to \mathbb{R} \mid u \text{ is continuous}\},$$
$$C^k(U) = \{u : U \to \mathbb{R} \mid D^\alpha u \in C(U) \ \text{ for } \ |\alpha| \le k\}.$$

When U is bounded

$C(\overline{U}) := \{u : \overline{U} \to \mathbb{R} \mid u \text{ is continuous}\}$,
$C^k(\overline{U}) := \{u \in C^k(U) \mid D^\alpha u \text{ has a continuous extension over } \overline{U} \text{ for each } |\alpha| \le k\}$

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1 Harmonic Functions

1.1 Definition and Mean value Properties

- Let us start this lecture with the definition of harmonic functions.

Definition

Let $U \subset \mathbb{R}^N$ be a domain. A function $u \in C^2(U)$ is called **harmonic function** on U or is said to be **harmonic** on U if u satisfies

$$\Delta u = 0 \quad \text{in } U.$$

- Let us give some characterization of the harmonic functions.

Definition

Let $U \subset \mathbb{R}^N$ be a domain and $u \in C(U)$.

(1) It is said that u satisfies the **second mean value property** if for any $\overline{B_r(x)} \subset U$

$$u(x) = \frac{1}{\omega_N r^{N-1}} \int_{\partial B_r(x)} u(y) d\sigma_y \tag{1.1}$$

holds, where $B_r(x) = \{y \in \mathbb{R}^N : |x-y| < r\}$ and ω_N denotes the surface area of the unit sphere in $\mathbb{R}^N$.

(2) It is said that u satisfies the **first mean value property** if for any $\overline{B_r(x)} \subset U$

$$u(x) = \frac{N}{\omega_N r^N} \int_{B_r(x)} u(y) dy \tag{1.2}$$

holds.

Remark:

(1) The above two conditions are equivalent. If $u \in C(U)$ satisfies the first mean value property, then for any $\overline{B_r(x)} \subset U$ and $0 \le \rho \le r$

$$u(x)\rho^{N-1} = \frac{1}{\omega_N} \int_{\partial B_\rho(x)} u(y) d\sigma_y.$$

Integrating both sides of the above identity from$\rho = 0$ to $\rho = r$ we obtain

$$u(x)\frac{r^N}{N} = \frac{1}{\omega_N} \int_{B_r(x)} u(y) dy.$$

If $u \in C(U)$ satisfies the second mean value property then for any $\overline{B_r(x)} \subset U$ we have

$$u(x)r^N = \frac{N}{\omega_N}\int_{B_r(x)} u(y)dy = \frac{N}{\omega_N}\int_0^r \int_{\partial B_\rho(x)} u(y)d\sigma_y d\rho.$$

Differentiating both sides of the above identity we obtain

$$Nu(x)r^{N-1} = \frac{N}{\omega_N}\int_{\partial B_r(x)} u(y)d\sigma_y.$$

(2) The identity (1.1) can be written as

$$u(x) = \frac{1}{\omega_N}\int_{\partial B_1(0)} u(x+rw)d\sigma_w$$

and the identity (1.2) can be written as

$$u(x) = \frac{N}{\omega_N}\int_{B_1(0)} u(x+rz)dz.$$

Theorem 1.1

If $u \in C^2(U)$ is a harmonic function, then u satisfies the first and second mean value properties.

Proof:

- We prove that u satisfies the first mean value property.
- Take any $\overline{B_r(x)} \subset U$ and define for $\rho \in (0, r)$

$$\phi(\rho) := \frac{1}{\omega_N}\int_{\partial B_1(0)} u(x+\rho w)d\sigma_w.$$

- Note that $u \in C^2(U)$ and $|Du|$ is bounded on $\partial B_\rho(x)$ so we can obtain

$$\begin{aligned}
\phi'(\rho) &= \frac{1}{\omega_N}\int_{\partial B_1(0)} \frac{d}{d\rho}u(x+\rho w)d\sigma_w \\
&= \frac{1}{\omega_N}\int_{\partial B_1(0)} Du(x+\rho w)\cdot w d\sigma_w \\
&= \frac{1}{\omega_N \rho^{N-1}}\int_{\partial B_\rho(x)} Du(y)\cdot \frac{y-x}{\rho}d\sigma_y \\
&= \frac{1}{\omega_N \rho^{N-1}}\int_{\partial B_\rho(x)} \frac{\partial u}{\partial \nu}d\sigma_y = \frac{1}{\omega_N \rho^{N-1}}\int_{B_\rho(x)} \Delta u(y)dy = 0.
\end{aligned}$$

- Therefore $\phi(\rho)$ is a constant function on $(0, r)$ and we obtain

$$u(x) = \lim_{\rho \downarrow 0} \phi(\rho) = \lim_{\rho \uparrow r} \phi(\rho) = \frac{1}{\omega_N r^{N-1}} \int_{\partial B_r(x)} u(y) d\sigma_y.$$

- The proof has been completed. □

Remark: Here we have used Gauss-Green formula: Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth $\partial\Omega$. For $u \in C^1(\overline{\Omega})$

$$\int_\Omega u_{x_i} dx = \int_{\partial\Omega} u \nu_i d\sigma_x,$$

where $\nu (= \nu(x)) = (\nu_1, \ldots, \nu_N)$ is the outward unit normal vector on $\partial\Omega$ (at $x \in \partial\Omega$). If $u \in C^2(\overline{\Omega})$, then we can replace u by u_{x_i} to obtain

$$\int_\Omega u_{x_i x_i} dx = \int_{\partial\Omega} u_{x_i} \nu_i d\sigma_x$$

and

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} Du \cdot \nu d\sigma_x = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma_x.$$

Theorem 1.2

If $u \in C^2(U)$ satisfies the mean-value property, then u is harmonic.

Proof:

- Suppose $\Delta u \not\equiv 0$. There exists a point $x \in U$ such that $\Delta u(x) > 0$ or $\Delta u(x) < 0$. We may assume the former, otherwise consider $-u$.

- Since $u \in C^2(U)$, there exists a ball $\overline{B_r(x)} \subset U$ such that $\Delta u > 0$ on $B_r(x)$.

- If we define $\phi(\rho)$ in the same way to the proof of Theorem 1.1, then from the mean-value property $\phi'(\rho)$ must be 0 for any $0 < \rho < r$.

- However from the computation in Theorem 1.1

$$\phi'(\rho) = \frac{1}{\omega_N \rho^{N-1}} \int_{B_\rho(x)} \Delta u(y) dy > 0,$$

  which is a contradiction. □

Remark: It can be shown that if $u \in C(U)$ satisfies the mean-value property, then $u \in C^2(U)$ and u is harmonic (see **[Evans]**, **[Han-Lin]**).

1.2 Maximum Principles

Theorem 1.3(Strong Maximum Principle)

Let $U \subset \mathbb{R}^N$ be a domain. Suppose that $u \in C(U)$ satisfies the mean-value property. If there exists $x_0 \in U$ such that $u(x_0) = \max_{x \in U} u(x)(=: M)$, then u is constant function in U.

Proof:

- Suppose that there exists $x_0 \in U$ such that $u(x_0) = \max_{x \in U} u(x)(=: M)$.
- Consider the following set

$$V := \{x \in U \mid u(x) = M\}.$$

  This set is relatively closed.

- Now we show that this set is open. By the mean-value property we have

$$M = u(x_0) = \frac{N}{\omega_N r^N} \int_{B_r(x_0)} u(x)dx \leq M$$

  for any $0 < r < \operatorname{dist}(x_0, \partial U)$. Therefore $u(x) \equiv M$ on $B_r(x_0)$. This means that V is an open set.

- Since U is connected $V = U$. $\square$

Remark:

(1) A set $V \subset U$ is called a relatively open set of U if for any $x_0 \in V$ there exists $r > 0$ such that $U_r(x_0) \cap U \subset V$. A set $F \subset U$ is called relatively closed if $U \setminus F$ is relatively open.

(2) We can also show that if there exists $x_0 \in U$ such that $u(x_0) = \min_{x \in U} u(x)(=: m)$ then $u \equiv m$ in U.

Colollary 1.4(Weak Maximum Principle)

Let $U \subset \mathbb{R}^N$ be a bounded domain. If $u \in C(\overline{U}) \cap C^2(U)$ is a harmonic function, then

$$\max_{x \in \overline{U}} u(x) = \max_{x \in \partial U} u(x)$$

and

$$\min_{x \in \overline{U}} u(x) = \min_{x \in \partial U} u(x).$$

Proof: We prove only the identity for the maximum.

- Since $\overline{U} \supset \partial U$, $\max_{x\in\overline{U}} u(x) \geq \max_{x\in\partial U} u(x)$ is obvious.
- Suppose that $\max_{x\in\overline{U}} u(x) > \max_{x\in\partial U} u(x)$ holds. Then there exists $x_0 \in U$ such that $u(x_0) = \max_{x\in U} u(x)$. However, from Theorem 1.3, u must be constant, which is a contradiction. □

Definition

Let $U \subset \mathbb{R}^N$ be a domain and $u \in C^2(U)$.

(1) u is said to be **superharmonic** if u satisfies $-\Delta u \geq 0$ in U.

(2) u is said to be **subharmonic** if u satisfies $-\Delta u \leq 0$ in U.

- In the view of the proof of Theorems 1.1 and 1.3 and Corollary 1.4 we can obtain the following results.

Proposition 1.5

Let $U \subset \mathbb{R}^N$ be a domain and $u \in C^2(U)$.

(1) If u is superharmonic, then for any $\overline{B_r(x)} \subset U$

$$u(x) \geq \frac{1}{\omega_N r^{N-1}} \int_{\partial B_r(x)} u(y) d\sigma_y, \quad u(x) \geq \frac{N}{\omega_N r^N} \int_{B_r(x)} u(y) dy.$$

(2) If u is subharmonic, then for any $\overline{B_r(x)} \subset U$

$$u(x) \leq \frac{1}{\omega_N r^{N-1}} \int_{\partial B_r(x)} u(y) d\sigma_y, \quad u(x) \leq \frac{N}{\omega_N r^N} \int_{B_r(x)} u(y) dy.$$

Proposition 1.6(Strong Maximum Principle)

Let $U \subset \mathbb{R}^N$ be a domain and $u \in C^2(U)$.

(1) Suppose that u is superharmonic. If there exists $x_0 \in U$ such that $u(x_0) = \min_{x\in U} u(x) (=: m)$, then $u \equiv m$ in U.

(2) Suppose that u is subharmonic. If there exists $x_0 \in U$ such that $u(x_0) = \max_{x\in U} u(x) (=: M)$, then $u \equiv M$ in U.

Corollary 1.7(Weak Maximum Principle)

Let $U \subset \mathbb{R}^N$ be a bounded domain and let $u \in C(\overline{U}) \cap C^2(U)$.

(1) If u is superharmonic, then

$$\min_{x \in \overline{U}} u(x) = \min_{x \in \partial U} u(x).$$

(2) If u is subharmonic, then

$$\max_{x \in \overline{U}} u(x) = \max_{x \in \partial U} u(x).$$

An application to the boundary value problem of the Poisson equations

- Let us consider the boundary value problem of the Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \tag{1.3}$$

where $U \subset \mathbb{R}^N$ is a bounded domain and, $f \in C(U)$ and $g \in C(\partial U)$ are given.

Proposition 1.8(Uniqueness)

Let u, $v \in C(\overline{U}) \cap C^2(U)$ be solutions to (1.3). Then $u = v$ in U.

Proof: Consider $w = u - v$. Then w is harmonic in U and $w = 0$ on ∂U. Therefore by Corollary 1.4, for any $y \in U$

$$0 = \min_{x \in \partial U} w(x) = \min_{x \in \overline{U}} w(x) \leq w(y) \leq \max_{x \in \overline{U}} w(x) = \max_{x \in \partial U} w(x) = 0.$$

Therefore $u \equiv v$ in U. $\square$

Proposition 1.9(Comparison principle)

Let u_1, $u_2 \in C(\overline{U}) \cap C^2(U)$ be solutions to (1.3) with $f = f_i$ and $g = g_i$ $(i = 1, 2)$ respectively. If $f_1 \geq f_2$ in U and $g_1 \geq g_2$ on ∂U, then $u_1 \geq u_2$ in U.

Proof: Consider $w = u_1 - u_2$. Then w is superharmonic and $w \geq 0$ on ∂U. By Corollary 1.7, for any $y \in U$

$$0 \leq \min_{x \in \partial U} w(x) = \min_{x \in \overline{U}} w(x) \leq w(y).$$

Therefore $u_1 \geq u_2$ in U. $\square$

Proposition 1.10(Strong Comparison principle)

Let u_1, $u_2 \in C(\overline{U}) \cap C^2(U)$ satisfy the same conditions of Proposition 1.9. Then $u_1 \equiv u_2$ in U or $u_1 > u_2$ in U holds.

Proof:

- Consider $w = u_1 - u_2$. Then w is superharmonic in U and $w \geq 0$ on ∂U. By Proposition 1.9, $w \geq 0$ in U.
- Suppose that $u_1(x_0) = u_2(x_0)$ at some point $x_0 \in U$. Then $w(x_0) = 0$.
- Since $w \geq 0$ in U, w takes its minimum at $x_0 \in U$.
- By Proposition 1.6, $w \equiv 0$ in U. Therefore if $u_1 \not\equiv u_2$ then $u_1 > u_2$ in U. $\square$

Remark:

- When U is **not** bounded, Propositions 1.8, 1.9 and 1.10 do not hold, in general.
- For example let $U = \{x \in \mathbb{R}^N : |x| > 1\}$ and let $u(x) = \log|x|$ when $N = 2$ and $u(x) = |x|^{2-N} - 1$ when $N \geq 3$. Then it is easily seen that u is a solution to (1.3) with $f = 0$, $g = 0$. However $v(x) \equiv 0$ is also solution to the same problem.

2 Maximum Principles for Elliptic Equations

2.1 Weak Maximum Principle

- Let $U \subset \mathbb{R}^N$ be a domain. For $u \in C^2(U)$, let us define the following differential operator:

$$\mathcal{L}u := -\sum_{i,j=1}^{N} a_{ij}(x)u_{x_i x_j}(x) + \sum_{i=1}^{N} b_i(x)u_{x_i}(x) + c(x)u, \tag{2.1}$$

where a_{ij}, b_i, $c \in C(U)$. Without loss of generality, we may assume that $a_{ij}(x) = a_{ji}(x)$ for $x \in U$. So we assume that $a_{ij}(x) = a_{ji}(x)$ for $x \in U$.

Definition

- The operator $\mathcal{L}$ defined in (2.1) is **elliptic** at $x \in U$ if the matrix $(a_{ij}(x))_{ij}$ is positive, that is, if $\lambda(x)$, $\Lambda(x)$ denote the minimum and maximum eigenvalues of $(a_{ij}(x))_{ij}$ respectively, then

$$0 < \lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2 \tag{2.2}$$

for any $\xi = {}^t(\xi_1, \cdots, \xi_N) \in \mathbb{R}^N \setminus \{0\}$. If $\lambda(x) > 0$ for $x \in U$, then $\mathcal{L}$ is said to be **elliptic** in U.

- If there exists a positive constant $\lambda_0 > 0$ such that $\lambda(x) \geq \lambda_0$ holds for $x \in U$, then $\mathcal{L}$ is said to be **strictly elliptic**.

- If $\Lambda(x)/\lambda(x)$ is bounded in U then $\mathcal{L}$ is called **uniformly elliptic**.

Remark: If a_{ij} are bounded functions and $\mathcal{L}$ is strictly elliptic, then $\mathcal{L}$ is uniformly elliptic. In this lecture, we assume that a_{ij}, b_j and c are bounded function and thus we assume that $\mathcal{L}$ is uniformly elliptic instead of strictly elliptic.

Example: If $a_{ij}(x) = \delta_{ij}$, $b_i(x) = 0$, $c(x) = 0$, then $\mathcal{L} = -\Delta$ and for $x \in U$ and $\xi = {}^t(\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j = \sum_{i=1}^{N}\sum_{j=1}^{N} \delta_{ij}\xi_i\xi_j = \sum_{i=1}^{N} \xi_i^2 = |\xi|^2$$

holds. Therefore $-\Delta$ is uniformly elliptic with $\Lambda(x) = \lambda(x) \equiv 1$.

Lemma 2.1

Suppose that operator $\mathcal{L}$ defined in (2.1) is elliptic in U and $c(x) \geq 0$ in U and $u \in C(\overline{U}) \cap C^2(U)$ satisfies $\mathcal{L}u < 0$. If u has a nonnegative maximum over $\overline{U}$, then u cannot attain this maximum at any point in U.

Proof:

- Suppose that $u(x_0) = \max_{x\in\overline{U}} u(x) \geq 0$ for some $x_0 \in U$.
- Then $u_{x_i}(x_0) = 0$ for $i = 1, \cdots, N$ and the matrix $B = (u_{x_i x_j}(x_0))_{ij}$ is nonpositive definite.
- Since the matrix $A = (a_{ij}(x_0))_{ij}$ is symmetric and positive definite, there exists an orthogonal matrix $T = (t_{ij})_{ij}$ so that

  $$ {}^tTAT = \mathrm{diag}(d_1, \ldots, d_N),\ {}^tTT = T{}^tT = E, $$

  with $d_i > 0$ $(i = 1, \ldots, N)$, that is,

  $$ \sum_{i=1}^N \sum_{j=1}^N t_{ik} a_{ij}(x_0) t_{jl} = d_k \delta_{kl},\ \sum_{k=1}^N t_{ik} t_{jk} = \delta_{ij}. \tag{2.3} $$

- Write $y = x_0 + {}^tT(x - x_0)$. Then $x - x_0 = T(y - x_0)$. We denote $u(x) = \tilde{u}(y) = \tilde{u}(x_0 + T(x - x_0))$.
- Then we have

  $$ \begin{aligned} u_{x_i}(x) &= \partial_{x_i}(\tilde{u}(y)) = \sum_{k=1}^N \tilde{u}_{y_k}(y)(y_k)_{x_i} = \sum_{k=1}^N \tilde{u}_{y_k}(y) t_{ik}, \\ u_{x_i x_j}(x) &= \sum_{k=1}^N \sum_{l=1}^N \tilde{u}_{y_k y_l}(y) t_{ik} t_{jl}. \end{aligned} $$

- Hence at the point x_0 by (2.3) we obtain

  $$ \begin{aligned} \sum_{i=1}^N \sum_{j=1}^N a_{ij}(x_0) u_{x_i x_j}(x_0) &= \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N a_{ij}(x_0) \tilde{u}_{y_k y_l}(x_0) t_{ik} t_{jl} \\ &= \sum_{k=1}^N \sum_{l=1}^N \tilde{u}_{y_k y_l}(x_0) \sum_{i=1}^N \sum_{j=1}^N a_{ij}(x_0) t_{ik} t_{jl} \\ &= \sum_{k=1}^N \sum_{l=1}^N \tilde{u}_{y_k y_l}(x_0) d_k \delta_{kl} = \sum_{k=1}^N \tilde{u}_{y_k y_k}(x_0) d_k. \end{aligned} $$

- Since $\tilde{u}$ takes its nonnegative maximum at x_0 we have $\tilde{u}_{y_k y_k}(x_0) \leq 0$ for $k = 1, \ldots, N$ and then

  $$ \sum_{i=1}^N \sum_{j=1}^N a_{ij}(x_0) u_{x_i x_j}(x_0) \leq 0,\ c(x_0)u(x_0) \geq 0. $$

- Therefore we obtain

$$\mathcal{L}u(x_0) = -\sum_{i,j=1}^{N} a_{ij}(x_0)u_{x_ix_j}(x_0) + \sum_{i=1}^{N} b_i(x_0)u_{x_i}(x_0) + c(x_0)u(x_0) \geq 0,$$

which is a contradiction to $\mathcal{L}u < 0$. $\square$

- From the proof of Lemma 2.1, it is easily obtain that the following corollary.

Corollary 2.2

Suppose that operator $\mathcal{L}$ defined in (2.1) is elliptic in U and $c(x) \equiv 0$ in U and $u \in C(\overline{U}) \cap C^2(U)$ satisfies $\mathcal{L}u < 0$. If u has a maximum over $\overline{U}$, then u cannot attain this maximum at any point in U.

- Now we state the weak maximum principle.

Theorem 2.3(Weak Maximum Principle)

Let $U \subset \mathbb{R}^N$ be a bounded domain. Suppose that operator $\mathcal{L}$ defined in (2.1) is uniformly elliptic in U and $c(x) \equiv 0$ in U and $u \in C^2(U) \cap C(\overline{U})$.

(1) If $\mathcal{L}u \leq 0$ in U, then

$$\max_{x\in\overline{U}} u(x) = \max_{x\in\partial U} u(x).$$

(2) If $\mathcal{L}u \geq 0$ in U, then

$$\min_{x\in\overline{U}} u(x) = \min_{x\in\partial U} u(x).$$

Proof:(1)

- We first note that by considering $\xi = \vec{e}_i$ (i-th fundamental vector) in (2.2)

$$a_{ii}(x) = \sum_{k,l=1}^{N} a_{kl}(x)\xi_k\xi_l \geq \lambda(x) > \lambda_0 > 0$$

- Let $u_\varepsilon(x) = u(x) + \varepsilon e^{\alpha x_1}$ for $x \in U$, where $\varepsilon > 0$ is any small constant and $\alpha > 0$ is a constant to be determined.

- Then by a direct computation implies that

$$\begin{aligned}\mathcal{L}u(x) &= \mathcal{L}u + \mathcal{L}(\varepsilon e^{\alpha x_1})\\ &= \mathcal{L}u + \varepsilon(-\alpha^2 a_{11}(x)e^{\alpha x_1} + \alpha b_1(x)e^{\alpha x_1})\\ &\leq \varepsilon\alpha e^{\alpha x_1}(-\alpha\lambda_0 + \|b_1\|_{L^\infty(U)}).\end{aligned}$$

Hence for $\alpha > \|b_1\|_{L^\infty(U)}\lambda_0^{-1}$ we see $\mathcal{L}(u_\varepsilon) < 0$.

- From Corollary 2.2 we obtain

$$\max_{x\in\overline{U}} u(x) \le \max_{x\in\overline{U}} u_\varepsilon = \max_{x\in\partial U} u_\varepsilon(x) \le \max_{x\in\partial U} u(x) + \varepsilon \max_{x\in\partial U} e^{\alpha x_1}.$$

  Here we remark that $0 \le \max_{x\in\partial U} e^{\alpha x_1} < \infty$ since ∂U is bounded.

- By letting $\varepsilon \to 0$ we con

$$\max_{x\in\overline{U}} u(x) \le \max_{x\in\partial U} u(x).$$

  On the other hand $\max_{x\in\partial U} u(x) \le \max_{x\in\overline{U}} u(x)$ is obvious. Therefore we obtain $\max_{x\in\overline{U}} u(x) = \max_{x\in\partial U} u(x)$.

(2) Consider $v = -u$ and note that

$$\max_{x\in\overline{U}}(-u) = -\min_{x\in\overline{U}} u(x) \quad \text{and} \quad \max_{x\in\partial U}(-u) = -\min_{x\in\partial U} u(x)$$

hold. □

- Now we will give the weak maximum principle for the case where $c \ge 0$. Now we define $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) = -\min\{u(x), 0\}$. We have $u(x) = u^+(x) - u^-(x)$ and $|u(x)| = u^+(x) + u^-(x)$.

Corollary 2.4

Let $U \subset \mathbb{R}^N$ be a bounded domain. Suppose that operator $\mathcal{L}$ defined in (2.1) is uniformly elliptic in U and $c(x) \ge 0$ in U and $u \in C(\overline{U}) \cap C^2(U)$.

(1) If $\mathcal{L}u \le 0$ in U, then

$$\max_{x\in\overline{U}} u(x) \le \max_{x\in\partial U} u^+(x).$$

(2) If $\mathcal{L}u \ge 0$ in U, then

$$\min_{x\in\overline{U}} u(x) \ge -\max_{x\in\partial U} u^-(x).$$

(3) In particular, if $\mathcal{L}u = 0$ in U, then

$$\max_{x\in\overline{U}} |u(x)| = \max_{x\in\partial U} |u(x)|.$$

Proof: (1)

- Let $\mathcal{L}u \leq 0$ in U. Consider set $V := \{x \in U : u(x) > 0\}$ and $\mathcal{K}u := \mathcal{L}u - c(x)u$.
- Then for $x \in V$
$$\mathcal{K}u = \mathcal{L}u - cu \leq -cu \leq 0.$$
Since the operator $\mathcal{K}$ has no zeroth-order term, by Theorem 2.1 we have
$$\max_{x\in\overline{V}} u(x) = \max_{x\in\partial V} u(x).$$
Noting $\partial V \subset \partial U \cup \{x \in U : u(x) = 0\}$ we obtain
$$\max_{x\in\overline{V}} u(x) = \max_{x\in\partial V} u(x) \leq \max_{x\in\partial U} u^+(x).$$
- In the case where $V \neq \emptyset$ we have $\max_{x\in\overline{U}} u(x) = \max_{x\in\overline{V}} u(x)$ and then we obtain $\max_{x\in\overline{U}} u(x) \leq \max_{x\in\partial U} u^+(x)$. Otherwise $u(x) \leq 0$ in $\overline{U}$ we obtain $\max_{x\in\overline{U}} u(x) \leq 0 = \max_{x\in\partial U} u^+(x)$.

(2) Consider $v = -u$ and note that $(-u)^+ = u^-$.

(3)

- It is enough to show that $\max_{x\in\overline{U}} |u(x)| \leq \max_{x\in\partial U} |u(x)|$.
- Since $-|u(x)| \leq u(x) \leq |u(x)|$ for $x \in \overline{U}$,
$$-\max_{x\in\partial U} |u(x)| \leq -\max_{x\in\partial U} u^-(x), \max_{x\in\partial U} u^+(x) \leq \max_{x\in\partial U} |u(x)|.$$
- By (1), (2) and the above observation we have for $y \in \overline{U}$.
$$\begin{aligned} -\max_{x\in\partial U} |u(x)| \leq -\max_{x\in\partial U} u^-(x) \leq \min_{x\in\overline{U}} u(x) \leq u(y) \\ \leq \max_{x\in\overline{U}} u(x) \leq \max_{x\in\partial U} u^+(x) \leq \max_{x\in\partial U} |u(x)| \end{aligned}$$
holds. The above inequality implies that $\max_{x\in\overline{U}} |u(x)| \leq \max_{x\in\partial U} |u(x)|$. □
- Finally, let us apply the maximum principle to the boundary value problem of the elliptic equation:
$$\begin{cases} \mathcal{L}u = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \tag{2.4}$$
where $U \subset \mathbb{R}^N$ is a bounded domain, $f \in C(U)$ and $g \in C(\partial U)$ are given functions.

- From the maximum principle we can obtain the following result:

Proposition 2.5

Let $U \subset \mathbb{R}^N$ be a bounded domain. Suppose that U is a bounded domain and operator $\mathcal{L}$ defined in (2.1) is strictly elliptic and $c(x) \geq 0$ in U. Let $f \in C(U)$ and $g \in C(\partial U)$ and let $u, v \in C(\overline{U}) \cap C^2(U)$ be solutions to (2.4). Then $u \equiv v$ in U.

Proof:

- Consider $w = u - v$. Then $\mathcal{L}w = 0$ in U and $w = 0$ on ∂U.
- By Corollary 2.4-(3)

$$\max_{x \in \overline{U}} |w(x)| = \max_{x \in \partial U} |w(x)| = 0.$$

- Therefore $w \equiv 0$ in U, that is, $u \equiv v$ in U. $\Box$

Proposition 2.6(Comparison principle)

Let $U \subset \mathbb{R}^N$ be a bounded domain. Suppose that $U \subset \mathbb{R}^N$ is a bounded domain and operator $\mathcal{L}$ defined in (2.1) is uniformly elliptic and $c(x) \geq 0$ in U. Let u_1, $u_2 \in C(\overline{U}) \cap C^2(U)$ be solutions to (2.4) with $f = f_i$ and $g = g_i$ $(i = 1, 2)$ respectively. If $f_1 \geq f_2$ in U and $g_1 \geq g_2$ on ∂U, then $u_1 \geq u_2$ in U.

Proof:

- Consider $w = u - v$. Then $\mathcal{L}w \geq 0$ in U and $w \geq 0$ on ∂U.
- By Corollary 2.4-(2) we obtain $\min_{x \in \overline{U}} w(x) = -\max_{x \in \partial U} w^-(x) = 0$. since $w^- = 0$ on ∂U.
- Therefore $w \geq 0$ in U, that is, $u \geq v$ in U. $\Box$

2.2 The Strong Maximum Principle

- In this subsection, we give the strong maximum principle for the general elliptic operator $\mathcal{L}$.
- To obtain the strong maximum principle the Hopf boundary point lemma, which will be given soon, is essential. To state the Hopf boundary point lemma, we give the notion the interior sphere condition.
- The domain $U \subset \mathbb{R}^N$ satisfies **interior sphere condition** at $x_0 \in \partial U$ if there exists $\overline{x} \in U$ and $r > 0$ such that $B_r(\overline{x}) \subset U$ and $x_0 \in \partial B_r(\overline{x})$.

Lemma 2.7(Hopf's Lemma)

Let $U \subset \mathbb{R}^N$ be a domain and operator $\mathcal{L}$ defined in (2.1) be uniformly elliptic. Assume $u \in C(\overline{U}) \cap C^2(U)$.

(1) In the case where $c \equiv 0$ in U, if $\mathcal{L}u \leq 0$ in U and there exists $x_0 \in \partial U$ such that U satisfies the interior sphere condition at x_0 and

$$u(x_0) > u(x) \text{ for all } x \in U, \tag{2.5}$$

then we have

$$\liminf_{t \to +0} \frac{u(x_0) - u(x_0 - t\nu)}{t} > 0,$$

where $\nu = \nu(x_0)$ denotes the outward unit normal vector of ∂U at x_0.

(2) In the case where $c(x) \geq 0$ in U, the same conclusion holds if $u(x_0) \geq 0$ and if $u(x_0) = 0$ the same conclusion holds irrespective of the sign of c.

Remark: If $\lim_{t \to +0} \dfrac{u(x_0) - u(x_0 - t\nu)}{t}$ exists, this limit is denoted by $\dfrac{\partial u}{\partial \nu}(x_0)$ and if $u \in C^1(\overline{U})$, this coincide with $Du(x_0) \cdot \nu$.

Proof:

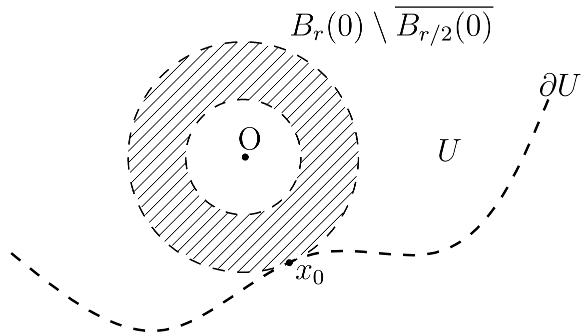
- We assume that $c \geq 0$ in U hold.
- By translation we assume that $\overline{x} = 0$ and $B_r(0) \subset U$ with $x_0 \in \partial B_r(0)$.
- Define

  $$v(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2} \quad \text{for} \quad x \in \overline{B_r(0)},$$

  where $\alpha > 0$ will be chosen later.
- By direct computation implies that

  $$\begin{aligned}\mathcal{L}v &= -\sum_{i,j=1}^N a_{ij}(x) v_{x_i x_j} + \sum_{i=1}^N b_i(x) v_{x_i} + c(x) v \\ &= e^{-\alpha|x|^2} \sum_{i,j=1}^N a_{ij}(x)(-4\alpha^2 x_i x_j + 2\alpha\delta_{ij}) \\ &\quad - e^{-\alpha|x|^2} \sum_{i=1}^N b_i(x) 2\alpha x_i + c(x)(e^{-\alpha|x|^2} - e^{-\alpha r^2}) \\ &\leq e^{-\alpha|x|^2}(-4\lambda_0\alpha^2|x|^2 + 2\alpha \mathrm{tr} A + 2\alpha|\boldsymbol{b}||x| + c),\end{aligned}$$

  where $A = (a_{ij}(x))_{ij}$, $\boldsymbol{b} = (b_1, \ldots, b_N)$ with $|\boldsymbol{b}| = \left(\sum_{i=1}^N |b_i|^2\right)^{1/2}$.



- We next consider $R := B_r(0) \setminus \overline{B_{r/2}(0)}$. Since $(r/2) < |x| < r$ for $x \in R$ we have

$$\mathcal{L}v \le e^{-\alpha|x|^2}(-\lambda_0\alpha^2 r^2 + 2\alpha \mathrm{tr} A + 2\alpha|\boldsymbol{b}|r + c).$$

  Hence $\mathcal{L}v \le 0$ on R provided $\alpha > 0$ is fixed large enough.

- By (2.5), $\inf_{x\in\partial B_{r/2}(0)}\{u(x_0) - u(x)\} > 0$. Hence there exists $\varepsilon > 0$ such that

$$u(x_0) \ge u(x) + \varepsilon v(x) \quad \text{for} \quad x \in \partial B_{r/2}(0).$$

  We also note that by (2.5) and $v = 0$ on $\partial B_r(0)$ we have

$$u(x_0) \ge u(x) + \varepsilon v(x) \quad \text{for} \quad x \in \partial B_r(0).$$

- Since $\mathcal{L}(u + \varepsilon v - u(x_0)) \le -cu(x_0) \le 0$ in R and $u + \varepsilon v - u(x_0) \le 0$ on ∂R we have

$$u(x) + \varepsilon v(x) \le u(x_0) \quad \text{for} \quad x \in R$$

  by the weak maximum principle.

- Since the outward normal vector ν of ∂U at x_0 coincides with the one of $B_r(0)$ at x_0, $\nu = x_0/r$. Hence for $t > 0$

$$\frac{u(x_0) - u(x_0 - t\nu)}{t} \ge -\varepsilon \frac{v(x_0) - v(x_0 - t\nu)}{t}$$

  and

$$\liminf_{t\to+0} \frac{u(x_0) - u(x_0 - t\nu)}{t} \ge -\varepsilon \lim_{t\to+0} \frac{v(x_0) - v(x_0 - t\nu)}{t} = -\varepsilon Dv(x_0) \cdot \frac{x_0}{r}.$$

- On the other hand by the direct calculation implies that $Dv(x_0) = -2\lambda x_0 e^{-\lambda r^2}$ and then $Dv(x_0) \cdot \frac{x_0}{r} = -2\lambda r e^{-\alpha r^2}$.

- Therefore

$$\liminf_{t\to+0} \frac{u(x_0) - u(x_0 - t\nu) - u(x_0)}{t} \ge 2\varepsilon\alpha r e^{-\alpha r^2} > 0$$

  and the proof has been completed. $\square$

Theorem 2.8(Strong Maximum Principle)

Suppose that U is a domain and operator $\mathcal{L}$ defined in (2.1) is uniformly elliptic in U and $c(x) \equiv 0$ in U and assume $u \in C(\overline{U}) \cap C^2(U)$.

(1) If $\mathcal{L}u \leq 0$ in U and u attains its maximum over $\overline{U}$ at an interior point (that is a point in U), then u is constant in U.

(2) If $\mathcal{L}u \geq 0$ in U and u attains its minimum over $\overline{U}$ at an interior point (that is a point in U), then u is constant in U.

Proof:

- Write $M := \max_{x\in\overline{U}} u(x)$, $C := \{x \in U : u(x) = M\}$ and $V := \{x \in U : u(x) < M\}$. We show $C = U$. Assume $C \neq U$.
- Suppose that there are two point x_0 and x_1 such that $x_0 \in V$ and $x_1 \in C$.
- Since U is a domain there exists a curve in U denoted by $\Gamma : \gamma = \gamma(t)$ $(0 \leq t \leq 1)$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.
- There exists a point $t_2 \in (0, 1]$ such that $u(\gamma(t_2)) = M$ and $u(\gamma(t)) < M$ for $t \in [0, t_2)$. Set $x_2 = \gamma(t_2)$.
- Since $x_2 \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x_2) \subset U$ and thus we can choose a point $y \in V$ such that $\mathrm{dist}(y, C) < \mathrm{dist}(y, \partial U)$(see Remark).
- Consider a largest ball $B_r(y)$ such that $B_r(y) \subset V$ (take $r = \mathrm{dist}(y, C)$). Then there exists a point $y_0 \in C$ with $y_0 \in \partial B_r(y)$.
- Since V satisfies the interior sphere condition at y_0 and $u(x) < u(y_0) = M$ in V, by Hopf's lemma we obtain

$$\frac{\partial u}{\partial \nu}(y_0) > 0$$

  However since u attains its maximum over $\overline{U}$ at $y_0 \in U$, $Du(y_0) = 0$ must hold. This is a contradiction. $\square$

Remark: How to choose $y \in V$ so that $\mathrm{dist}(y, C) < \mathrm{dist}(y, \partial U)$.

- Take $t_3 \in (0, t_2)$ close to t_2 so that $y = \gamma(t_3) \in B_{\varepsilon/4}(x_2)$
- Then we have $\mathrm{dist}(y, C) \leq |y - x_2| \leq \varepsilon/4$.
- On the other hand if $x \in \partial U$ then $x \notin B_\varepsilon(x_2)$. Therefore

$$|x - y| \geq |x - x_2| - |y - x_2| \geq \frac{3}{4}\varepsilon$$

  and $\mathrm{dist}(t, U) \geq (3/4)\varepsilon > (1/4)\varepsilon \geq \mathrm{dist}(y, C)$.

Theorem 2.9(The Strong Maximum Principle with $c \geq 0$)

Suppose that U is a domain and operator $\mathcal{L}$ defined in (2.1) is uniformly elliptic in U and $c(x) \geq 0$ in U and assume $u \in C(\overline{U}) \cap C^2(U)$.

(1) If $\mathcal{L}u \leq 0$ in U and u attains a nonnegative maximum over $\overline{U}$ at an interior point (that is a point in U), then u is constant in U.

(2) If $\mathcal{L}u \geq 0$ in U and u attains a nonpositive minimum over $\overline{U}$ at an interior point (that is a point in U), then u is constant in U.

Proof: By using (2) of Lemma 2.7, we can same line as the proof of Theorem 2.8. □

2.3 The Phragmèn-Lindelöf Principle

- The weak maximum principle hold when U is a **bounded** domain. When U is **not** bounded Theorem 2.3 and Corollary 2.4 does not hold.

Example:

- Let $U = (0, \pi) \times \mathbb{R} \subset \mathbb{R}^2$. Note that U is not bounded.
- Let $u(x, y) = \sin x \cosh 2y$. Then u satisfies

$$\begin{cases} -\Delta u + 3u = 0 & \text{in } \ U, \\ u = 0 & \text{on } \ \partial U. \end{cases} \tag{2.6}$$

However $\max\limits_{(x,y)\in U} |u(x, y)| = \max\limits_{(x,y)\in \partial U} |u(x, y)|$ does not hold since $u(x, y) > 0$ for $(x, y) \in U$.

- For unbounded U, we have to impose the condition on the growth of the function at infinity.

Theorem 2.9(The Phragmèn-Lindelöf Principle)

Suppose that U is an unbounded domain and operator $\mathcal{L}$ defined in (2.1) is uniformly elliptic in U and $c(x) \equiv 0$ in U and assume that there exists a positive function $\phi \in C(\overline{U}) \cap C^2(U)$ such that

$$\mathcal{L}\phi \geq 0, \quad \lim_{|x|\to\infty, x\in U} \phi(x) = \infty$$

If $u \in C(\overline{U}) \cap C^2(U)$ satisfies

$$\mathcal{L}u \leq 0, \quad \liminf_{A\to\infty} \sup_{\phi(x)=A, x\in U} \frac{u(x)}{\phi(x)} \leq 0$$

and $u \leq 0$ on ∂U, then $u \leq 0$ in U.

Proof:

- Fix $y \in U$. We will prove for any $\varepsilon > 0$ $u(y) \le \varepsilon\phi(y)$. Choose any $\varepsilon > 0$.

- By the assumption there exists $A_\varepsilon > 0$ such that for $A \ge A_\varepsilon$

$$\sup_{\phi(x)=A, x\in U} \frac{u(x)}{\phi(x)} \le \varepsilon \tag{2.7}$$

  holds. We choose $A \ge A_\varepsilon$ so that $\phi(y) < A$ holds.

- Set $w(x) := \dfrac{u(x)}{\phi(x)}$. By the direct calculation we obtain

$$\begin{aligned} 0 \ge \mathcal{L}u &= \mathcal{L}(\phi w) && (2.8)\\ &= \phi\mathcal{L}w - c(x)\phi w - \sum_{i=1}^{N}\sum_{j=1}^{N} 2a_{ij}\phi_{x_i}w_{x_j} + (\mathcal{L}\phi)w && (2.9)\end{aligned}$$

  If we define the elliptic operator

$$\tilde{\mathcal{L}}v := -\sum_{i=1}^{N}\sum_{j=1}^{N}\phi(x)a_{ij}(x)w_{x_ix_j} - \sum_{j=1}^{N}\tilde{b}_j(x)w_{x_j} + \tilde{c}(x)v$$

  where

$$\tilde{b}_j(x) = \sum_{i=1}^{N} 2a_{ij}(x)\phi_{x_i}(x), \quad \tilde{c}(x) = \mathcal{L}\phi(x) \ge 0,$$

  then it holds that $\tilde{\mathcal{L}}w \le 0$.

- Consider open set $U_A = \{x \in U : \phi(x) < A\}$. Assumption $\phi(x) \to \infty$ (as $|x| \to \infty$) implies that U_A is bounded. Since $\phi(x) > 0$ in $\overline{U}$ there exists $a_0 > 0$ such that $\phi(x) \ge a_0$ for $x \in U_A$. Therefore $\tilde{\mathcal{L}}$ is strictly elliptic in U_A.

- By the weak maximum principle (Corollary 2.4-(1)) we obtain

$$w(z) \le \max_{x\in\partial U_A} w^+ = \max_{x\in\partial U_A} \max\left\{0, \frac{u(x)}{\phi(x)}\right\} \le \varepsilon \quad \text{for} \quad z \in U_A.$$

  and thus $u(y) \le \varepsilon\phi(y)$. Letting $\varepsilon \to 0$ we obtain $u(y) \le 0$. Since $y \in U$ is arbitrary, we obtain $u(x) \le 0$ for $x \in U$. $\square$

Example:

- Let us consider again the example given in the begging of this subsection. In this example function $\phi(x, y) = \cosh \sqrt{3}y$ satisfies (2.7). Therefore if u satisfies $u \leq 0$ for $(x, y) \in \partial U$ and

$$\liminf_{A \to \infty} \sup_{\cosh \sqrt{3}y = A} \frac{u(x, y)}{\cosh \sqrt{3}y} = \liminf_{B \to \infty} \sup_{x \in (0,\pi), y = B} \frac{u(x, y)}{\cosh \sqrt{3}y} \leq 0,$$

  then $u \leq 0$ in U. However $u(x, y) = \sin x \cosh 2y$ does not satisfy this condition.

- We note that if condition (2.7) holds, then any **bounded** function satisfies (2.8).