

# 2024年度 解析Ⅲ 演習(担当:松澤 寛) レジメ

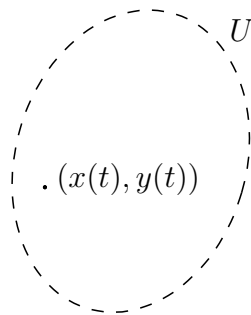
## No.4 合成関数の微分

### まとめ1 (合成関数)

- 関数  $z = f(x, y)$  は点  $(a, b)$  の近傍  $U$  で定義されているとする.
- 関数  $x(t), y(t)$  はある開区間  $I$  で定義されているとする.
- 点  $(x(t), y(t))$  が

$$(x(t), y(t)) \in U \quad (\forall t \in I)$$

であれば合成関数  $z = z(t) = f(x(t), y(t))$  ( $t \in I$ ) が定義される.



### まとめ2 (合成関数の微分)

上の状況下でさらに  $z = f(x, y)$  は点  $(a, b)$  で全微分可能,  $x = x(t), y = y(t)$  は  $t = c$  で微分可能で  $x(c) = a, y(c) = b$  ならば  $z = z(t) = f(x(t), y(t))$  は  $t = c$  で微分可能で

$$\begin{aligned} \frac{dz}{dt}(c) &= \frac{\partial z}{\partial x}(a, b) \frac{dx}{dt}(c) + \frac{\partial z}{\partial y}(a, b) \frac{dy}{dt}(c) \\ [z'(c) &= f_x(a, b)x'(c) + f_y(a, b)y'(c)] \end{aligned}$$

### 導関数の場合

$z = f(x, y)$  が  $U$  の各点で全微分可能,  $x = x(t), y = y(t)$  がすべての  $t \in I$  で微分可能であるとき,  $z = z(t) = f(x(t), y(t))$  はすべての  $t \in I$  で微分可能で

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) \end{aligned}$$

### まとめ3 (合成関数の(偏)微分法)

- $z = f(x, y)$  は点  $(a, b)$  で全微分可能
- $x = x(u, v), y = y(u, v)$  は点  $(c, d)$  で全(偏)微分可能で  $a = x(c, d), b = y(c, d)$

であるならば  $z = z(u, v) = f(x(u, v), y(u, v))$  は点  $(c, d)$  で全(偏)微分可能で

$$\frac{\partial z}{\partial u}(c, d) = \frac{\partial z}{\partial x}(a, b) \frac{\partial x}{\partial u}(c, d) + \frac{\partial z}{\partial y}(a, b) \frac{\partial y}{\partial u}(c, d)$$
$$[z_u(c, d) = f_x(a, b)x_u(c, d) + f_y(a, b)y_u(c, d)]$$

$$\frac{\partial z}{\partial v}(c, d) = \frac{\partial z}{\partial x}(a, b) \frac{\partial x}{\partial v}(c, d) + \frac{\partial z}{\partial y}(a, b) \frac{\partial y}{\partial v}(c, d)$$
$$[z_v(c, d) = f_x(a, b)x_v(c, d) + f_y(a, b)y_v(c, d)]$$

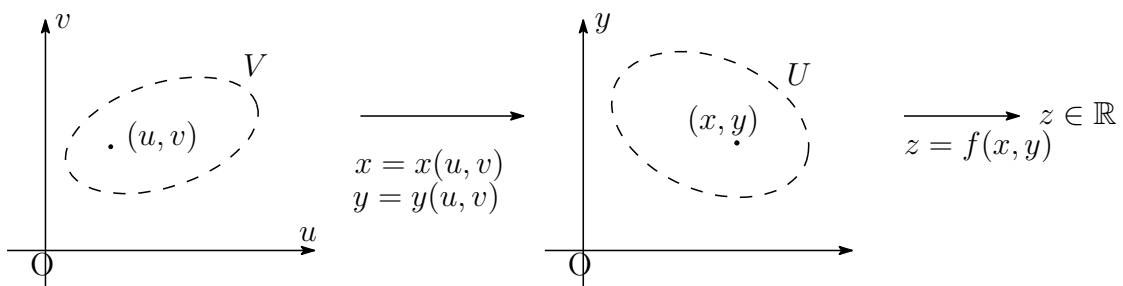
### 偏導関数の場合

- $z = f(x, y)$  は  $U$  の各点で全微分可能
- $x = x(u, v), y = y(u, v)$  は  $uv$  平面の領域  $V$  の各点で全微分可能で,  $(x(u, v), y(u, v)) \in U$   
( $\forall (u, v) \in V$ )

とすると,  $z = z(u, v) = f(x(u, v), y(u, v))$  は  $V$  の各点  $(u, v)$  で全微分可能で

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$



**演習 1**

$z = \log \frac{x}{y}$ ,  $x = e^t + e^{-t}$ ,  $y = e^t - e^{-t}$  のとき  $\frac{dz}{dt}$  を求めよ.

**解**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

ここで  $z = \log x - \log y$  より

$$\frac{\partial z}{\partial x} = \frac{1}{x}, \quad \frac{\partial z}{\partial y} = -\frac{1}{y}$$

また

$$\frac{dx}{dt} = e^t - e^{-t}, \quad \frac{dy}{dt} = e^t + e^{-t}$$

よって

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{x}(e^t - e^{-t}) - \frac{1}{y}(e^t + e^{-t}) \\ &= \frac{1}{e^t + e^{-t}}(e^t - e^{-t}) - \frac{1}{e^t - e^{-t}}(e^t + e^{-t}) \\ &= \frac{(e^t - e^{-t})^2 - (e^t + e^{-t})^2}{(e^t + e^{-t})(e^t - e^{-t})} \\ &= -\frac{4}{e^{2t} - e^{-2t}} // \end{aligned}$$

**演習 2** (講義でも扱うかもしれませんが大切な問題なので)

$z = f(x, y)$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$  の時  $\frac{\partial z}{\partial r}$ ,  $\frac{\partial z}{\partial \theta}$ ,  $\frac{\partial^2 z}{\partial r^2}$ ,  $\frac{\partial^2 z}{\partial \theta^2}$  を求めよ. ただし,  $f(x, y)$  は第 2 次偏導関数まで連続とする.

**解**  $z = f(r \cos \theta, r \sin \theta)$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x} (-r \sin \theta) + \frac{\partial z}{\partial y} (r \cos \theta)$$

ここで  $\frac{\partial z}{\partial x}$ ,  $\frac{\partial z}{\partial y}$  はそれぞれ  $f_x(r \cos \theta, r \sin \theta)$ ,  $f_y(r \cos \theta, r \sin \theta)$  であり, それぞれ何かある  $x, y$  の 2 変数関数の  $x, y$  に  $r \cos \theta, r \sin \theta$  が代入されていることに注意.

$$\begin{aligned} \frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} \right) \cos \theta + \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial y} \right) \sin \theta \quad \leftarrow \cos \theta, \sin \theta \text{ は定数扱い} \\ &= \left( \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial r} \right) \cos \theta + \left( \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial r} \right) \sin \theta \\ &= \left( \frac{\partial^2 z}{\partial x^2} \cos \theta + \frac{\partial^2 z}{\partial y \partial x} \sin \theta \right) \cos \theta + \left( \frac{\partial^2 z}{\partial x \partial y} \cos \theta + \frac{\partial^2 z}{\partial y^2} \sin \theta \right) \sin \theta \quad \leftarrow \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} \\ &= \frac{\partial^2 z}{\partial x^2} \cos^2 \theta + 2 \frac{\partial^2 z}{\partial y \partial x} \cos \theta \sin \theta + \frac{\partial^2 z}{\partial y^2} \sin^2 \theta \end{aligned}$$

次に  $\frac{\partial^2 z}{\partial \theta^2}$  を計算する.

$$\left[ \begin{array}{l} \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} (-r \sin \theta) + \frac{\partial z}{\partial y} (r \cos \theta) \quad \leftarrow \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \text{ にも } \theta \text{ があるので積の微分} \\ f_x(r \cos \theta, r \sin \theta) \quad f_y(r \cos \theta, r \sin \theta) \end{array} \right]$$

$$\begin{aligned} \frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial}{\partial \theta} \left( \frac{\partial z}{\partial x} \right) (-r \sin \theta) + \frac{\partial z}{\partial x} \frac{\partial}{\partial \theta} (-r \sin \theta) + \frac{\partial}{\partial \theta} \left( \frac{\partial z}{\partial y} \right) (r \cos \theta) + \frac{\partial z}{\partial y} \frac{\partial}{\partial \theta} (r \cos \theta) \\ &= \left( \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial \theta} \right) (-r \sin \theta) + \frac{\partial z}{\partial x} (-r \cos \theta) \\ &\quad + \left( \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial \theta} \right) + \frac{\partial z}{\partial y} (-r \sin \theta) \\ &= \left( \frac{\partial^2 z}{\partial x^2} (-r \sin \theta) + \frac{\partial^2 z}{\partial y \partial x} (r \cos \theta) \right) (-r \sin \theta) + \frac{\partial z}{\partial x} (-r \cos \theta) \\ &\quad + \left( \frac{\partial^2 z}{\partial x \partial y} (-r \sin \theta) + \frac{\partial^2 z}{\partial y^2} (r \cos \theta) \right) + \frac{\partial z}{\partial y} (-r \sin \theta) \quad \leftarrow \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} \\ &= r^2 \frac{\partial^2 z}{\partial x^2} \sin^2 \theta - 2r^2 \frac{\partial^2 z}{\partial x \partial y} \cos \theta \sin \theta + r^2 \frac{\partial^2 z}{\partial y^2} \cos^2 \theta - r \frac{\partial z}{\partial x} \cos \theta - r \frac{\partial z}{\partial y} \sin \theta \quad // \end{aligned}$$