Advanced Topic in Modern Mathematical Sciences
Lecturer: Hiroshi Matsuzawa
Self Check Sheet No.01

Student No. ____________ Name________________

1. Write the definition that $u \in C^2(U)$ is a harmonic function.

2. Write the definitions that $u \in C(U)$ satisfies the first and the second mean value property.

3. Write the statement of the strong maximum principle for harmonic functions.

4. Write the definitions that $u \in C^2(U)$ is a superharmonic function and subharmonic function.

5. Referring to the lecture note, give the proof of Propositions 1.5 and 1.6 for the superharmonic functions.

6. Write the general form of the elliptic operator given in the lecture and write the definition that the operator is strictly elliptic and uniformly elliptic.

7. Write the statement of the weak maximum principle for the case where the elliptic operator does not have the 0th order term.

8. Write the statement of Hopf's Lemma.

9. Write the statement of the strong maximum principle for the case where the elliptic operator does not have the 0th order term.

10. Let $I = (a, b)$ be a bounded interval. Consider $\mathcal{L}u = -u'' + g(x)u'$ with a bounded continuous function g on $[a, b]$. Let $u \in C^2(a, b) \cap C[a, b]$ satisfies $\mathcal{L}u < 0$. Prove that u cannot take its maximum over $[a, b]$ at any point in (a, b).

11. (**One-dimensional maximum prinnciple**) Suppose that $u \in C^2(a, b) \cap C[a, b]$ satisfies $\mathcal{L}u = -u'' + g(x)u' \leq 0$ and $u(c) = \max_{x\in[a,b]} u(x) =: M$ for some $c \in (a, b)$. Show u must be constant by answering the following questions

(1) Consider $z(x) = e^{\alpha(x-c)} - 1$. Compute $\mathcal{L}z = -z'' + g(z)z'$ and prove that $\mathcal{L}z < 0$ in (a, b) for some large α.

(2) Suppose that u is not a constant function. There exists $d \in (a, b)$ such that $u(d) < M$. We may assume $d > c$. Then prove that there exists $\varepsilon > 0$ such that $w(x) = u(x)+\varepsilon z(x)$ satisfies $w(a) < M$, $w(d) < M$.

(3) By using the result of **9** get a contradiction.

(4) How about when $d < c$?

12. (**One-dimensional Hopf's lemma**) Suppose that $u \in C^2(a, b) \cap C[a, b]$ satisfies $\mathcal{L}u = -u'' + g(x)u' \leq 0$. Prove if $u(a) = \max\limits_{x\in[a,b]} u(x) =: M$ and u has a one-side derivative $u'(a) = \lim\limits_{h\to+0} \dfrac{u(a + h) - u(a)}{h}$, then $u'(a) < 0$ or $u \equiv M$(Hint: Suppose that $u \not\equiv M$, there exists $d \in (a, b)$ such that $u(d) < M$. Consider $z(x) = e^{\alpha(x-a)} - 1$ and considering $w = u+\varepsilon z$ on (a, d)).